MATH 4362 PARTIAL DIFFERENTIAL EQUATIONS

§1 WHAT ARE PDEs?

TERMINOLOGY

① A DIFFERENTIAL EQUATION (DE) is an equation relating the derivatives of a function, u

② A SOLUTION of a DE is a function that satisfies the equation.

③ The ORDER of a DE is the order of the highest derivative in the equation.

* DEs often arise in the mathematical modeling of physical phenomena.

④ EQUILIBRIUM EQUATIONS model physical systems that do not change with time.

So $\quad u = u(\vec{x}), \quad \vec{x} = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$

SPATIAL VARIABLES.

⑤ DYNAMICAL EQUATIONS model processes that change ~~~~ with time

So $\quad u = u(t) \quad$ or $\quad u = u(t, \vec{x})$.

Most DEs arising in physics are 1st or 2nd order

(6) ORDINARY DEs (ODEs) involve a function of 1 variable,

$u = u(t)$ DYNAMICAL ODE

$u = u(x)$ EQUILIBRIUM ODE

EXS

(a) ODE $\frac{du}{dt} = au$ 1ST ORDER

SOLN $u(t) = Ce^{at}$ for some constant, C.

Models exponential growth $(a > 0)$ or decay $(a < 0)$.

(b) ODE $\ddot{u} + ku = 0$ 2nd Order

$\left(\cdot = \frac{d}{dt} \right)$

If $k > 0$

SOLN $u(t) = A\sin(\sqrt{k}\,t) + B\cos(\sqrt{k}\,t)$

for some constants A, B.

Models simple harmonic motion.

GOALS OF COURSE: ① Where do PDEs come from?
② How do we solve them?
③ What properties do the solutions have? ③

⑦ PARTIAL DEs (PDEs) involve a function of 2 or more variables

EG $u = u(x, y)$ for EQUILIBRIUM PDES
$u = u(t, x)$ for DYNAMICAL PDES

EXAMPLES WE WILL STUDY

ⓐ TRANSPORT EQUATION for $u = u(t, x)$:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

1ST ORDER DYNAMICAL PDE

Models transport of a substance in a fluid

ONE SOLUTION: $u(t, x) = e^{-(x-t)^2}$

SOLNS LIKE THIS THAT ARE SMOOTH (SUFFICIENTLY DIFFERENTIABLE) ARE CALLED CLASSICAL SOLUTIONS

CHECK

$$\frac{\partial u}{\partial t} = e^{-(x-t)^2}\left(-2(x-t)\cdot(-1)\right)$$
$$= 2(x-t)\, e^{-(x-t)^2}$$

$$\frac{\partial u}{\partial x} = e^{-(x-t)^2}\left(-2(x-t)\right)$$
$$= -2(x-t)\, e^{-(x-t)^2}$$

So $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ ✓

QN Can you guess another solution?

LAPLACIAN

ON $\mathbb{R}$: $\quad \Delta = \dfrac{\partial^2}{\partial x^2}$

ON $\mathbb{R}^2$: $\quad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

ON $\mathbb{R}^3$: $\quad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

(b) $\boxed{\text{HEAT EQUATION ON } \mathbb{R}^n}$ for $u = u(t, \vec{x})$, $\vec{x} \in \mathbb{R}^n$ is

$$\frac{\partial u}{\partial t} = \Delta u$$

2ND ORDER DYNAMICAL PDE.

Models diffusion (spread) of heat

ONE SOLUTION: $u(t, x) = t + \frac{1}{2}x^2$ ON $\mathbb{R}$

CHECK

$$\frac{\partial u}{\partial t} = 1, \qquad \frac{\partial u}{\partial x} = x$$

$$\frac{\partial^2 u}{\partial x^2} = 1$$

So $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} = \Delta u$.

ANOTHER SOLUTION (The "Fundamental Solution")

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \quad \text{ON } \mathbb{R}$$

$$u(t,x) = \frac{1}{2\sqrt{\pi}} t^{-1/2} e^{-\frac{x^2}{4}t^{-1}}$$

$$\frac{\partial u}{\partial t} = \frac{1}{2\sqrt{\pi}}\left[-\frac{1}{2} t^{-3/2} e^{-\frac{x^2}{4t}} + t^{-1/2}\left(-\frac{x^2}{4}(-t^{-2})\right) e^{-\frac{x^2}{4t}}\right]$$

$$= \frac{1}{2\sqrt{\pi}}\left[-\frac{1}{2} t^{-3/2} + \frac{x^2}{4} t^{-5/2}\right] e^{-\frac{x^2}{4t}}$$

$$= \frac{1}{2\sqrt{\pi}}\left[-2t + x^2\right] \frac{1}{4} t^{-5/2} e^{-\frac{x^2}{4t}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{\pi}} t^{-1/2}\left(-\frac{x}{2t}\right) e^{-\frac{x^2}{4t}}$$

$$= -\frac{1}{4\sqrt{\pi}} t^{-3/2} x e^{-x^2/4t}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{4\sqrt{\pi}} t^{-3/2}\left[e^{-x^2/4t} - x \cdot \frac{x}{2t} e^{-x^2/4t}\right]$$

$$= -\frac{1}{4\sqrt{\pi}}\left[2t - x^2\right] \frac{1}{2} t^{-5/2} e^{-\frac{x^2}{4t}}$$

$$= \frac{1}{2\sqrt{\pi}}\left[-2t + x^2\right] \frac{1}{4} t^{-5/2} e^{-\frac{x^2}{4t}}$$

$$= \frac{\partial u}{\partial t} \quad ✓$$

$\boxed{u \text{ IS A CLASSICAL SOLN ON } \{(t,x) \mid t>0,\ x \in \mathbb{R}\}}$

BIG QUESTION What happens as $t \to 0^+$?

(c) WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \qquad \text{for } u = u(t, \vec{x}), \quad \vec{x} \in \mathbb{R}^n$$

2nd ORDER, DYNAMICAL

SPECIAL CASE: $u_{tt} = u_{xx}$ for $u = u(t, x)$, $x \in \mathbb{R}$.

Models wave motion in many contexts:
- Sound (Pressure) waves
- Light waves
- Vibrations of strings and drums.

SOLN in SPECIAL CASE $x \in \mathbb{R}$:

Let f, g be any twice-differentiable functions of a single variable, ξ.

EX $f(\xi) = \cos \xi$, $\quad g(\xi) = e^{-\xi}$

LET $\boxed{u(t, x) = f(x-t) + g(x+t)}$

Then u solves $u_{tt} = u_{xx}$

CHECK

$$u_t \overset{CR}{=} f'(x-t)\frac{\partial}{\partial t}(x-t) + g'(x+t)\frac{\partial}{\partial t}(x+t) = -f'(x-t) + g'(x+t)$$

So $u_{tt} = f''(x-t) + g''(x+t)$
$= \ldots$
$= u_{xx}$ ✓

(d) LAPLACE EQN : $\Delta u = 0$
POISSON'S EQN : $\Delta u = f$, $f = f(\vec{x})$ GIVEN

Poisson's Eqn models

- Steady-state (equilibrium) distribution of heat
- Equilibrium displacement of membranes (drums)
- Gravitational potential due to a distribution of masses.
- Electric potential due to a distribution of charges
- Potential functions for steady-state fluid flow

A SOLN TO $u_{xx} + u_{yy} = 0$ ON $\mathbb{R}^2$:

$u(x,y) = \log(x^2+y^2)$ "FUNDAMENTAL SOL^n".

$$u_x = \frac{2x}{x^2+y^2}$$

$$u_{xx} = \frac{2(x^2+y^2) - 2x.2x}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

So

$$u_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$ by swapping roles of x, y

UPSHOT

$$u_{xx} + u_{yy} = 0 \checkmark$$

u IS A CLASSICAL SOLUTION ON $\{(x,y) \in \mathbb{R}^2 / (x,y) \neq (0,0)\}$

BIG QN What happens at $(x,y) = (0,0)$??

So far all PDEs have been linear.

(2) 2 NONLINEAR PDES !

$$u_t + u u_x = 0$$ NON-LINEAR TRANSPORT

$$u_t + u u_x = u_{xx}$$ BURGER'S EQN

— Simplified model of fluid dynamics

INITIAL CONDITIONS + BOUNDARY CONDITIONS

DEs typically have infinitely many solutions.

EX (a) ODE $\ddot{u} + u = 0$ has solutions

$$u(t) = A \sin t + B \cos t \quad \text{for } A, B \in \mathbb{R}$$

(b) PDE $u_{tt} + u_{xx} = 0$ has sol^{ns}

$$u(t, x) = f(x - t) + g(x + t)$$

for functions f, g.

To model a particular physical system we need to impose some additional conditions to ensure that there is only ONE solution.

- For Dynamical ODEs we impose initial conditions

EX

$$\begin{cases} \ddot{u} + u = 0 \\ u(0) = \alpha \\ \dot{u}(0) = \beta \end{cases} \quad \text{for given } \alpha, \beta.$$

Solve for A, B above in terms of α, β to get unique sol^n.

- For equilibrium ODEs we impose boundary conditions

EX $u = u(x)$

$$\begin{cases} u'' + u = 0 & \text{on } (0,1) \qquad ' = \frac{d}{dx} \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$$

Once again, solve for A, B above in terms of α, β.

- For Equilibrium PDEs we need to specify the domain on which the solution u is defined on and specify conditions on the boundary of that domain

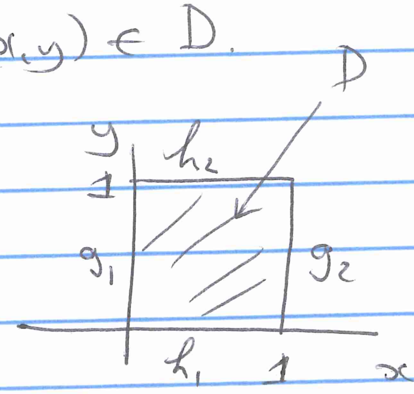
EX

$\boxed{\text{POISSON BVP}}$

(a)

$$\begin{cases} u_{xx} + u_{yy} = f(x,y) & (x,y) \in D. \\ u(0, y) = g_1(y) \\ u(1, y) = g_2(y) \\ u(x, 0) = h_1(x) \\ u(x, 1) = h_2(x) \end{cases}$$

DIRICHLET BCs



- For Dynamical PDE's we also need to specify initial conditions (at time $t=0$), one for each time derivative in the eqn.

EX $\boxed{\text{WAVE EQN IBVP}}$

$$\begin{cases} u_{tt} + u_{xx} = 0 & \text{on} \quad 0 < x < 1 \\ u(0, x) = v(x) & \\ u_t(0, x) = w(x) & \text{IC's} \\ u(t, 0) = g(t) & \\ u(t, 1) = h(t) & \text{BC's} \end{cases}$$

where v, w, g, h are given functions.

LINEAR AND NONLINEAR EQNS

EXS of Linear Differential Operators:

ⓐ $L = \partial_t + \partial_x = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$

ⓑ $L = \frac{\partial^2}{\partial x^2}$ or $L = \Delta$

Generally a Linear Differential Operator is a linear combination of partial derivative operators with coefficients that are either

constants

or functions of t and $\vec{x}$.

A HOMOGENEOUS LINEAR DE is one of form

$$L[u] = 0$$

for a linear differential operator L.

EG

(a) Transport Eqn

$$(\partial_t + \partial_x)[u] = 0$$

OR $\quad u_t + u_x = 0$

(b) Laplace eqn

$$\Delta[u] = 0$$

or $\quad u_{xx} + u_{yy} = 0 \quad$ on $\mathbb{R}^2$.

RIGOROUS DEFN (Use for hwk + exams)

A differential operator, L, is linear if

(a) $L[u+v] = L[u] + L[v]$
(b) $L[cu] = c\,L[u]$

for all smooth functions u, v and constants $c \in \mathbb{R}$.

EX (a) $L = \frac{\partial^2}{\partial x^2}$ is linear as

$$\frac{\partial^2}{\partial x^2}[cu+v] = c\frac{\partial^2}{\partial x^2}[u] + \frac{\partial^2}{\partial x^2}[v]$$

(b) $L[u] = u\frac{\partial u}{\partial x}$ is NOT linear as

$$L[2u] = 2u\frac{\partial(2u)}{\partial x} = 4u\frac{\partial u}{\partial x}$$
$$= 4\,L[u]$$
$$\neq 2\,L[u].$$

THM [SUPERPOSITION PRINCIPLE]

If $u_1, \ldots, u_k$ are solutions to a Homogeneous LINEAR DE $L[u]=0$
Then so is

$$v = \sum_{j=1}^{k} c_j u_j \qquad c_j \in \mathbb{R}.$$

PF

$$L[v] = L\left[\sum_{j=1}^{k} c_j u_j\right]$$

$$= \sum_{j=1}^{k} c_j L[u_j] \qquad \text{as } L \text{ is linear}$$

$$= \sum c_j 0$$

$$= 0. \qquad \square$$

CONSEQUENCE

The set of all solutions of $L[u]=0$ forms a VECTOR SPACE.

NOTE For PDE's this ~~solution~~ space is usually INFINITE DIMENSIONAL, and ~~we~~ we need to represent the general ~~solutions~~ solution as an infinite series of functions

– This leads to notion of FOURIER SERIES.

DEF An INHOMOGENEOUS Linear DE is one of the form

$$L[v] = f$$

where L is a LDO and f is a given f^n.

THM
Let v_* be a PARTICULAR SOLUTION of $L[v]=f$
Then the general solution of $L[v]=f$ is of the form $v = v_* + u$
where u solves $L[u] = 0$.

PF
ⓐ IF $L[u] = 0$ and $v = v_* + u$ then

$$L[v] = L[v_* + u] = L[v_*] + L[u] = f + 0 = f$$

ⓑ IF v is any sol^n of $L[v] = f$ Then set $u = v - v_*$.
We have $L[u] = L[v - v_*] = L[v] - L(v_*) = f - f = 0$.

THM (SUPERPOSITION FOR INHOMOGENEOUS DE's)

Let $f_1 - f_k$ be given function.

Suppose $\quad f = \sum_{j=1}^{k} c_j f_j$

If $v_1 - v_k$ solve $\quad L[v_j] = f_j$

Then $\quad v = \sum_{j=1}^{k} c_j v_j$ solves $L[v] = f$

PF

$$L[v] = \sum_{j=1}^{k} c_j L[v_j] = \sum_{j=1}^{k} c_j f_j = f \quad \square$$