Math 4362 Homework #7

1. Jon Bell, Lecture #8: page 9, Question 4

2. Jon Bell, Lecture #8: page 9, Question 6

3. Consider the initial value problem

$$u_t = Du_{xx}, \qquad x \in \mathbb{R}, t > 0 \tag{1}$$
$$u(0,x) = \tfrac{1}{2}(H(x+1) - H(1-x)), \tag{2}$$

   where H is the Heavyside function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{3}$$

   Derive a formula for the solution of this IVP in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy. \tag{4}$$

4. Solve the initial value problem

$$u_t = Du_{xx}, \qquad x \in \mathbb{R}, t > 0 \tag{5}$$
$$u(0,x) = H(x)e^{-x}. \tag{6}$$

5. (a) Let $u(t,x)$ satisfy

$$u_t = u_{xx}, \qquad \text{for } 0 < x < 1, t > 0 \tag{7}$$
$$u(t,0) = u(t,1) = 0 \qquad \text{for } t \geq 0 \tag{8}$$
$$u(0,x) = f(x), \qquad \text{for } 0 \leq x \leq 1 \tag{9}$$

   for some continuous function, f, on $[0,1]$. Show that for any $T \geq 0$,

$$\int_0^1 (u(T,x))^2\, dx \leq \int_0^1 (f(x))^2\, dx. \tag{10}$$

   Hint: use the identity $2u(u_t - u_{xx}) = (u^2)_t - (uu_x)_x + 2(u_x)^2$.

   (b) Use (10) to derive a uniquness theorem for the initial-boundary value problem in (a).

6. Let $u(t,x)$ be a solution of

$$u_t - u_{xx} = f(t,x), \tag{11}$$

   in the region $R = \{(t,x) : 0 < x < L,\ 0 < t < T\}$. Show that if u is continuous on the closed rectangle, $\overline{R}$, and $f(t,x) < 0$ for all $(t,x) \in \overline{R}$, then the maximum of u is attained where $t = 0$ or $x = 0$ or $x = L$.

Recommended Problems [Not to be handed in]

1. Jon Bell, Lecture #8: page 1, Question 1
2. Jon Bell, Lecture #8: page 5, Question 2
3. Jon Bell, Lecture #8: page 9, Question 3