INVARIANCE OF LEBESGUE MEASURE

THM

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transform^n

and let $A \subseteq \mathbb{R}^n$.

~~Thm~~ If A is Lebesgue measurable

Then $T(A)$ is Lebesgue measurable

and

$$\lambda(T(A)) = |\det T| \; \lambda(A).$$

In particular Lebesgue measure is invariant under translations and rotations (orthogonal matrices)

PF

OMIT

PF IDEA

Any $T: \mathbb{R}^n \xrightarrow{LT} \mathbb{R}^n$ is a composition of elementary transf^{ns} of form

$$[T] = \begin{bmatrix} c & \\ & I_{n-1} \end{bmatrix}$$

or

$$[T] = \begin{bmatrix} 1 & c & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Can Assume T is elem.

Let J = UNIT CUBE and $\rho := \dfrac{\lambda(T(J))}{\lambda(J)}$

Pave open $G = \bigcup_{k=1}^{\infty} J_k$ $\quad J_k = z_k + t_k J$

— Trans^n + Dilation of J

$$\lambda(TG) = \lambda\left(\bigcup_{k=1}^{\infty} TJ_k\right) = \sum_{k=1}^{\infty} \lambda(TJ_k)$$

$$= \rho \sum_{k=1} \lambda(J_k) = \rho\, \lambda(G)$$

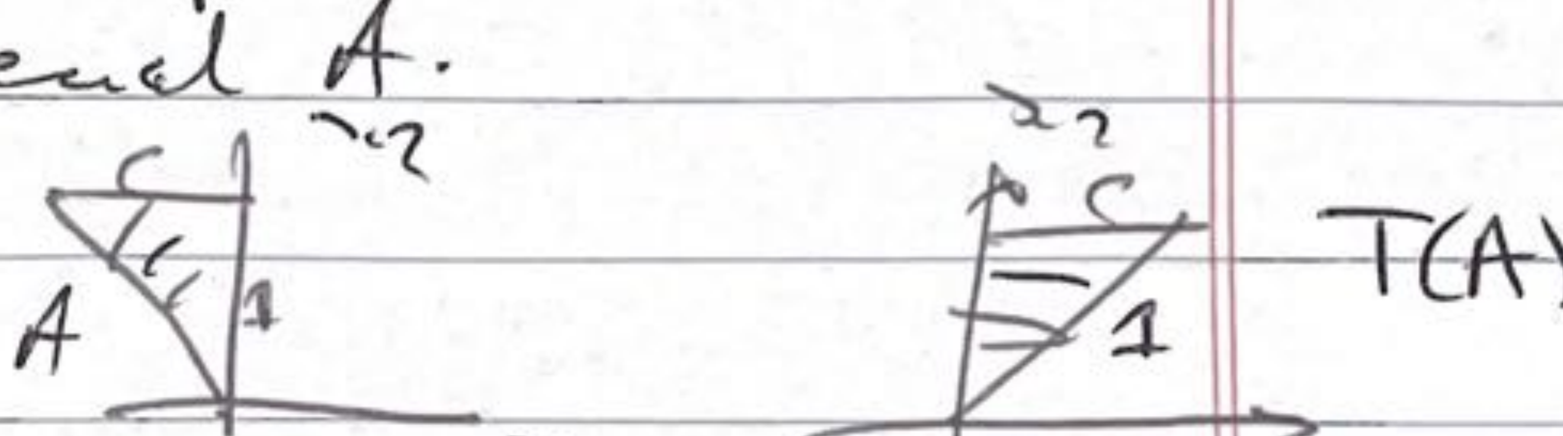
NOTE T is CTS so preserves openness ~~etc~~

So can get same $\lambda(TA) = \rho\, \lambda(A)$ $\forall A \in \mathcal{L}$

Must show $\rho = |\det T|$ (ρ is indep^t of A)

To do so can choose special A.

For 2nd case above pick



$TA = MA$ where

$M = \begin{bmatrix} 1 & \\ & I_{n-1} \end{bmatrix}$ so get $\rho = |\det T| = 1$.

For 1st case $A = [0,1]^n$ is OK.