THM 8 (COUNTABLE ADDITIVITY)

Suppose A_k is Leb M'ble (with $\lambda(A_k) < \infty$) $\forall k$

Let $A = \bigcup_{k=1}^{\infty} A_k$

Suppose $\lambda^*(A) < \infty$.

Then A is Leb Mble and

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

If in add^n A_k's are disjt then

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

PF

1 CASE A_k's DISJT

$$\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^*(A_k) \text{ by } *3$$

$$= \sum_{k=1}^{\infty} \lambda(A_k) \quad \text{as } A_k \text{ M'ble}$$

$$= \sum_{k=1}^{\infty} \lambda_*(A_k)$$

\#4 USES DISJOINTNESS

$$\leq \lambda_*(A) \leq \lambda^*(A)$$

So equality holds and

$$\lambda_*(A) = \lambda^*(A)$$

So A is Leb M'ble and by equalities above

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

② General Case

Can write A as a disjt union:

Define

$$B_1 = A_1$$
$$B_2 = A_2 \sim A_1$$

$$B_k = A_k \sim (A_1 \cup \dots \cup A_{k-1}) \quad \forall k$$

By Cor 7 B_k is Leb M'ble $\forall k$

Also ~~UB~~ B_k are disjt, $B_k \subset A_k$, $A = \bigcup_{k=1}^{\infty} B_k$

So by (1) A is Leb M'ble and

$$\lambda(A) = \sum_k \lambda(B_k) \underset{B_k \subset A_k}{\leq} \sum_k \lambda(A_k)$$

$\Box$

STAGE 6 ARBITRARY MEASURABLE SETS

DEF 9

$$\mathcal{L}_0 = \{ A \subset \mathbb{R}^n / \lambda^*(A) < \infty \text{ and } \lambda_*(A) = \lambda^*(A) \}$$

is set of Leb Meas'ble sets $\bar{c}$ finite measure

DEF 10

$$\mathcal{L} = \{ A \subseteq \mathbb{R}^n / A \cap M \in \mathcal{L}^0 \quad \forall M \in \mathcal{L}_0 \}$$

Define for $A \in \mathcal{L}$

$$\lambda(A) = \sup \{ \lambda(A \cap M) / M \in \mathcal{L}_0 \}$$

CONSISTENCY CHECK BETWEEN STAGES 5/6

PROP 11 Let $A \subset \mathbb{R}^n$, Assume $\lambda^*(A) < \infty$.

Then

$$A \in \mathcal{L}_0 \iff A \in \mathcal{L}$$

AND

$$\lambda_{\mathcal{L}_0}(A) = \lambda_{\mathcal{L}}(A)$$

$\boxed{\Rightarrow}$ Let $A \in \mathcal{L}_0$

Then $\forall M \in \mathcal{L}_0$, $A \cap M \in \mathcal{L}_0$ by COR 7.
So $A \in \mathcal{L}$ by defn.

$\boxed{\Leftarrow}$ Suppose $A \in \mathcal{L}$.

Ball $B(0,k) \in \mathcal{L}_0$

Let $A_k = A \cap B(0,k) \in \mathcal{L}_0$ by def of $\mathcal{L}$

Then $A = \bigcup_{k=1}^{\infty} A_k$ and we know $\lambda^*(A) < \infty$

So by Thm 8, $A \in \mathcal{L}_0$ holds.

NEXT Let $A \in \mathcal{L}$ ($A \in \mathcal{L}_0$ too)

$\lambda_{\mathcal{L}}(A) = \sup\{\lambda_{\mathcal{L}_0}(A \cap M) \mid M \in \mathcal{L}_0\}$

NOW $A \cap M \subset A$

So $\lambda_{\mathcal{L}_0}(A \cap M) \leq \underbrace{\lambda_{\mathcal{L}_0}(A)}_{UB}$

So $\underset{LUB}{\lambda_{\mathcal{L}}(A)} \leq \lambda_{\mathcal{L}_0}(A)$.

ALT Choose $M = A \in \mathcal{L}_0$. So $\underset{SUP}{\lambda_{\mathcal{L}}(A)} \geq \lambda_{\mathcal{L}_0}(A)$ $\square$

NOTE The example in "Why Lebesgue" Lecture that involved Axiom of Choice can be adapted to construct a set $E \subseteq \mathbb{R}^n$ That is not Lebesgue Measurable [J, 2.B]

PROPERTIES OF LEBESGUE MEASURE

PFS SIMILAR SPIRIT TO ONES ALREADY DONE. See [J. 2. B]

(M1) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$

(M2) Countable Unions + Countable Intersections of Leb M'ble sets are Leb M'ble

(M3) $A, B \in \mathcal{L} \Rightarrow A \sim B \in \mathcal{L}$

(M4) If A_k are M'ble Then

$$\lambda\left(\bigcup_k A_k\right) \leq \sum_k \lambda(A_k)$$

If A_k also disj't

$$\lambda\left(\bigcup_k A_k\right) = \sum_k \lambda(A_k)$$

(M5) If $A_k \in \mathcal{L}$ and $A_1 \subset A_2 \subset \cdots$ Then

$$\lambda\left(\bigcup_k A_k\right) = \lim_{k \to \infty} \lambda(A_k)$$

M16 If $A_k \in \mathcal{L}$, $A_1 \supset A_2 \supset \ldots$, $\lambda(A_1) < \infty$.

Then

$$\lambda\left(\bigcap_k A_k\right) = \lim_{k \to \infty} \lambda(A_k)$$

M7 All open and closed sets are Mble

M8 If $\lambda^*(A) = 0$ Then A is Mble, $\lambda(A) = 0$

M9 $A \in \mathcal{L} \iff \forall \varepsilon > 0 \ \exists$ Closed F, Open G :

$$F \subset A \subset G$$

$$\lambda(G \sim F) < \varepsilon$$

M10 If $A \in \mathcal{L}$ Then $\lambda^*(A) = \lambda(A) = \lambda_*(A)$

M11 If $A \subset B$, B Mble Then

$$\lambda^*(A) + \lambda_*(B \sim A) = \lambda(B)$$

M12 CARATHEODORY COND N (Only Involves λ^*)
BUT Must Check $\underline{\underline{\forall}}\, E \subset \mathbb{R}^n$

$A \in \mathcal{L} \iff \forall E \subset \mathbb{R}^n$

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^c)$$

INVARIANCE OF LEBESGUE MEASURE

THM

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transform^n

and let $A \subseteq \mathbb{R}^n$.

~~Then~~ If A is Lebesgue measurable

Then $T(A)$ is Lebesgue measurable

and

$$\lambda(T(A)) = |\det T| \, \lambda(A).$$

In particular Lebesgue measure is

invariant under translations and

rotations (orthogonal matrices)

PF

OMIT