LECTURE 15 — INNER + OUTER MEASURE, MEASURABLE SETS [J, #2]

DEF 1 Let $A \subseteq \mathbb{R}^n$ be ARBITRARY. Define

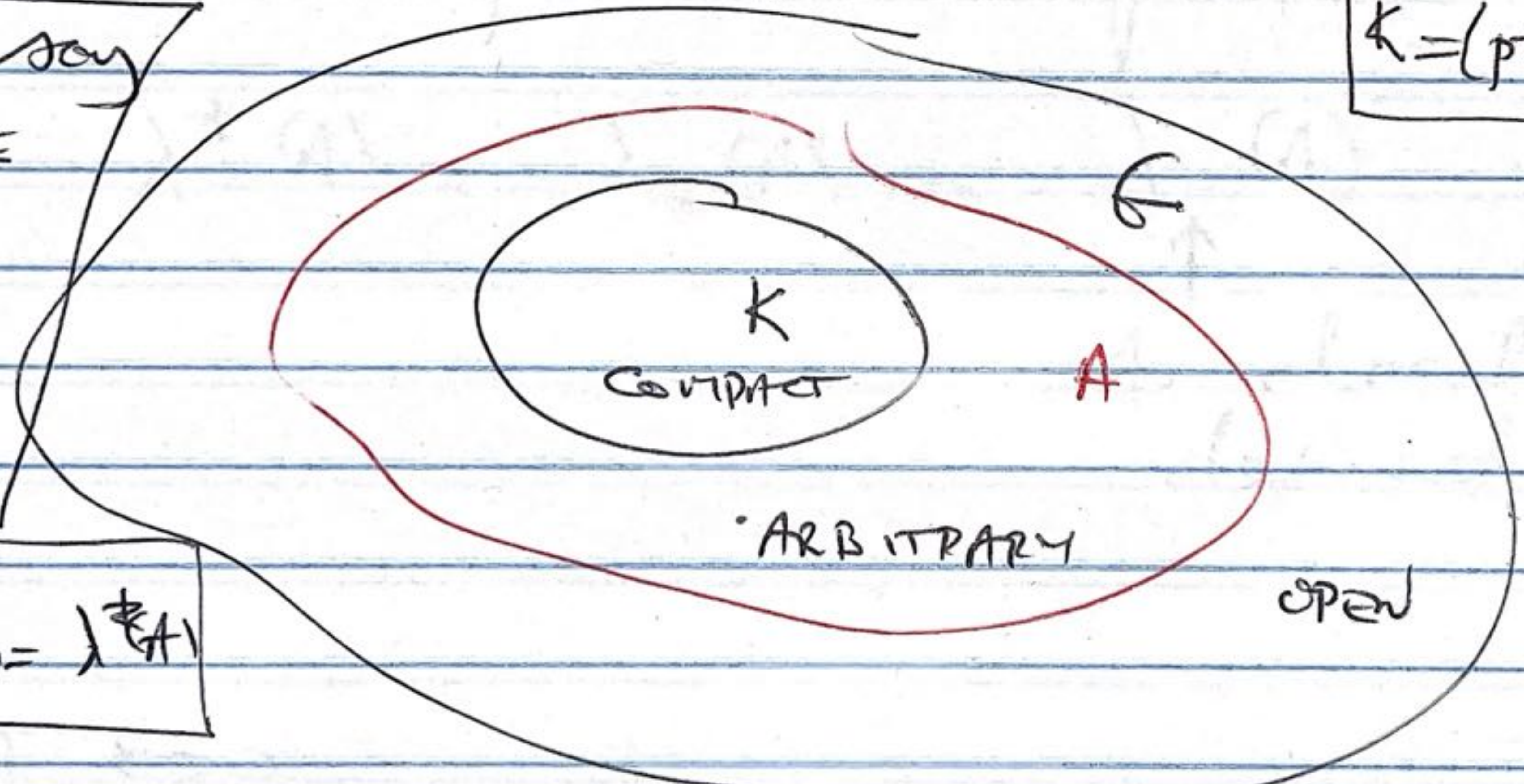
OUTER MEASURE of $A = \lambda^*(A) = \inf\{\lambda(G) \,/\, A \subseteq G,\ G \text{ open}\}$

$\boxed{A \text{ bounded} \Rightarrow \lambda^*(A) < \infty}$

INNER MEASURE of $A = \lambda_*(A) = \sup\{\lambda(K) \,/\, K \subseteq A,\ K \text{ compact}\}$

$\boxed{K = \{pt\} \Rightarrow \lambda_*(A) \geq 0}$

IDEA Later say A is LEBESGUE MEASURABLE IF $\lambda_*(A) = \lambda^*(A)$ and define $\lambda(A) = \lambda_*(A) = \lambda^*(A)$



PROP 2 Let $A \subseteq \mathbb{R}^n$ be arbitrary

*1. $0 \leq \lambda_*(A) \leq \lambda^*(A)$

A bounded $\Rightarrow \lambda^*(A) < \infty$.

*2. $A \subseteq B \Rightarrow \lambda_*(A) \leq \lambda_*(B)$

$\lambda^*(A) \leq \lambda^*(B)$

*3 $\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$

*4 IF A_k's are pairwise disjoint Then

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(A_k)$$

*5 IF A is open or compact Then

$$\lambda^*(A) = \lambda_*(A) = \lambda(A)$$

As already defined.

PF

LET $\varepsilon > 0$

*1 By properties of LUB & GLB

$\exists K \subset A :$ $\underbrace{\lambda_*(A)}_{LUB} - \varepsilon < \lambda(K)$

$\exists G \supset A :$ $\lambda^*(A) + \varepsilon > \lambda(G)$.

ALSO $K \subset G$ So $\lambda(K) < \lambda(G)$ by defn $\lambda(K)$

So $\lambda_*(A) - \varepsilon < \lambda^*(A) + \varepsilon \quad \forall \varepsilon > 0$

B

*2 Similar to proofs for open cpt sets.

*3 Let $\varepsilon > 0$

By defn of λ^* $\exists$ open $G_k \supset A_k$:

$$\lambda(G_k) < \underbrace{\underset{GLB}{\lambda^*(A_k)} + \varepsilon 2^{-k}}_{\text{NOT LB.}}$$

So by (OS) $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} G_k$ open

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$$

INF

$$\leq \sum_{k=1}^{\infty} \lambda(G_k)$$

$$< \sum_{k=1}^{\infty} \lambda^*(A_k) + \varepsilon \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \sum_{k=1}^{\infty} \lambda^*(A_k) + \varepsilon$$

4. Let $K_j \subset A_j$ be compact, $j = 1, \dots, N$.

Since $\{A_k\}_{k=1}^{\infty}$ are disjoint so are $\{K_j\}_{j=1}^{\infty}$.

So by def^n of λ_* as a sup and fact

$$\underbrace{\bigcup_{k=1}^{N} K_k}_{\text{COMPACT}} \subseteq \bigcup_{k=1}^{\infty} A_k$$

we have

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \lambda\left(\bigcup_{k=1}^{N} K_k\right) \overset{C4}{=} \sum_{k=1}^{N} \lambda(K_k) \quad (\dagger)$$

LET $\varepsilon > 0$

Since $\lambda_*(A_k) - \frac{\varepsilon}{2^k}$ is not an upper bound

$\exists\, K_k$:

$$\lambda_*(A_k) - \frac{\varepsilon}{2^k} \leq \lambda(K_k)$$

So by $(\dagger)$

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{N} \lambda(K_k) \geq \sum_{k=1}^{N} \lambda_*(A_k) - \varepsilon \sum_{k=1}^{N} \left(\frac{1}{2}\right)^k$$

$$\geq \sum_{k=1}^{N} \lambda_*(A_k) - \varepsilon \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \sum_{k=1}^{N} \lambda_*(A_k) - \varepsilon$$

SINCE TRUE $\forall \varepsilon, N$ we get $\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(A_k)$

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#5

$\boxed{\text{CASE } A \text{ OPEN}}$

$\Lambda_A = \{ \lambda(G) / G \supseteq A, \ G \text{ open} \}$

$\lambda(A) \in \Lambda_G$ as A is open

(a) $\boxed{\underset{GLB}{\lambda^*(A)} \leq \lambda(A)}$ as $G = A$ is open

(b) $\boxed{\lambda^*(A) \geq \lambda(A)}$ as OR $\forall G \supseteq A, \quad \lambda(G) \geq \lambda(A)$

So $\lambda(A)$ is LB for $\Lambda_A = \{ \lambda(G) / G \supseteq A, \ G \text{ open} \}$

So $\underset{GLB}{\lambda^*(A)} \geq \underset{LB}{\lambda(A)}$.

(c) $\boxed{\lambda(A) \leq \lambda_*(A)}$

$\lambda_*(A) = \sup \{ \lambda(K) / K \subset A \text{ Cpt} \}$

$\lambda(A) \underset{A \text{ open}}{=} \sup \{ \lambda(P) / P \subset A \text{ Spec Poly} \}$

Let $\varepsilon > 0$

Choose $P \subset A$ So that $\lambda(A) - \varepsilon < \lambda(P)$

Since P is Cpt $\quad \lambda(P) \leq \lambda_*(A)$ ← LUB

So $\lambda(A) - \varepsilon < \lambda_*(A) \implies \lambda(A) \leq \lambda_*(A)$

(d) $\lambda_*(A) \leq \lambda^*(A) \overset{(a)}{\leq} \lambda(A)$

$\boxed{\text{CASE } A \text{ COMPACT}}$ Similar.

STAGE 5 SETS WITH FINITE OUTER MEASURE

DEF Let

$$\mathcal{L}_0 = \{ A \subseteq \mathbb{R}^n \,/\, \lambda_*(A) = \lambda^*(A) < \infty \}$$

We say A is Lebesgue measurable if $A \in \mathcal{L}_0$.

and define Lebesgue measure of A to be

$$\lambda(A) := \lambda_*(A) = \lambda^*(A).$$

NOTE We impose condition $\lambda^*(A) < \infty$ because otherwise any set with $\lambda_*(A) = \infty$ would be Lebesgue measurable.

PROP 4

(1) If A is compact then $A \in \mathcal{L}_0$

(2) If A is open with $\lambda(A) < \infty$ Then $A \in \mathcal{L}_0$.

PF See ~~?~~ Π

LEMMA 5

Let A, B be Lebesgue m'ble, $A \cap B = \emptyset$.

Then $A \cup B$ is Leb m'ble and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

PF

We ~~know~~ $\lambda_*(A \cup B) \leq \lambda^*(A \cup B)$.

We must show $\lambda^*(A \cup B) \leq \lambda_*(A \cup B)$

Well

$$\lambda^*(A \cup B) \overset{*3}{\leq} \lambda^*(A) + \lambda^*(B)$$

$$= \lambda(A) + \lambda(B) \quad \text{as } A, B \text{ are Leb m'ble}$$

$$= \lambda_*(A) + \lambda_*(B) \quad \text{"}$$

$$\overset{*4}{\leq} \lambda_*(A \cup B)$$

So $\lambda_*(A \cup B) = \lambda^*(A \cup B) = \lambda(A) + \lambda(B)$ ✓

□

APPROX^{N} THM 6

A is Leb M'ble $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists$ compact K, open G:

$$K \subset A \subset G$$

and $\lambda(G \sim K) < \varepsilon$.

ANALOGOUS TO f RI $\Rightarrow \overset{\forall \varepsilon > 0}{\exists P}: S(f, P) - s(f, P) < \varepsilon$

PF

$\boxed{\Rightarrow}$ LET $\varepsilon > 0$.

By def^{n} outer measure, as

$$\lambda^*(A) = \inf\{\lambda(G) \mid A \subset G,\ G \text{ open}\},$$

$\exists G$: $A \subset G$ and $\lambda(G) < \lambda^*(A) + \varepsilon/2$
$= \lambda(A) + \varepsilon/2$

By def^{n} inner measure, as

$$\lambda_*(A) = \sup\{\lambda(K) \mid K \subset A,\ K \text{ cpt}\}$$

$\exists K$: $K \subset A$ and $\lambda(K) > \lambda^*(A) - \varepsilon/2$
$= \lambda(A) - \varepsilon/2$

By Lemma 5

$$\lambda(G \sim K) = \lambda(G) - \lambda(K) < (\lambda(A) + \varepsilon/2) - (\lambda(A) - \varepsilon/2)$$
$$= \varepsilon$$

$\boxed{\Leftarrow}$ We know $\lambda_*(A) \le \lambda^*(A)$.

Must show $\lambda^*(A) \le \lambda_*(A)$.

By assump^n $\forall \varepsilon > 0$

$$\begin{aligned}\lambda^*(A) &\le \lambda(G) \\ &= \lambda(K) + \lambda(G \sim K) \\ &= \lambda(K) + \varepsilon \\ &\le \lambda_*(A) + \varepsilon\end{aligned}$$

So $\lambda^*(A) \le \lambda_*(A)$ ✓

Cor 7

Let A, B be Leb M'ble.
Then so are $A \cup B$, $A \cap B$, $A \sim B$.

Pf

(1) $\boxed{A \sim B}$

Let $\varepsilon > 0$
By Thm 6 have

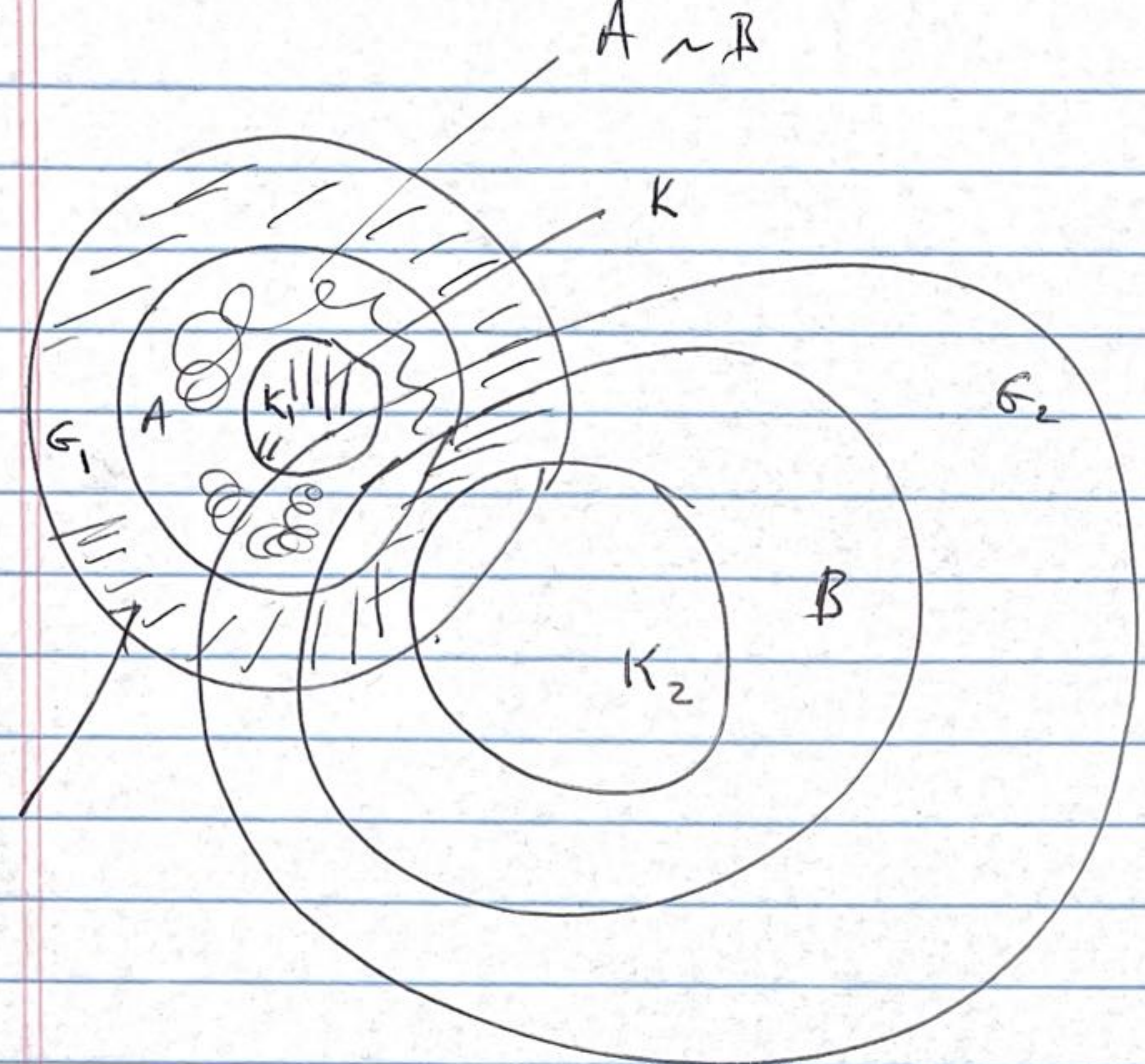
$K_1 \subset A \subset G_1$ $\quad \lambda(G_1 \sim K_1) < \varepsilon/2$

$K_2 \subset B \subset G_2$ $\quad \lambda(G_2 \sim K_2) < \varepsilon/2$

Define $K = K_1 \sim G_2$ COMPACT $[K \cap F$ is compact$]$ A∩M CLOSED

$G = G_1 \sim K_2$ OPEN $[G_1 \cap G_2$ is open$]$

$G_2 = \mathbb{R}^n \sim K_2$



UHV $\quad K \subset A \sim B \subset G$.

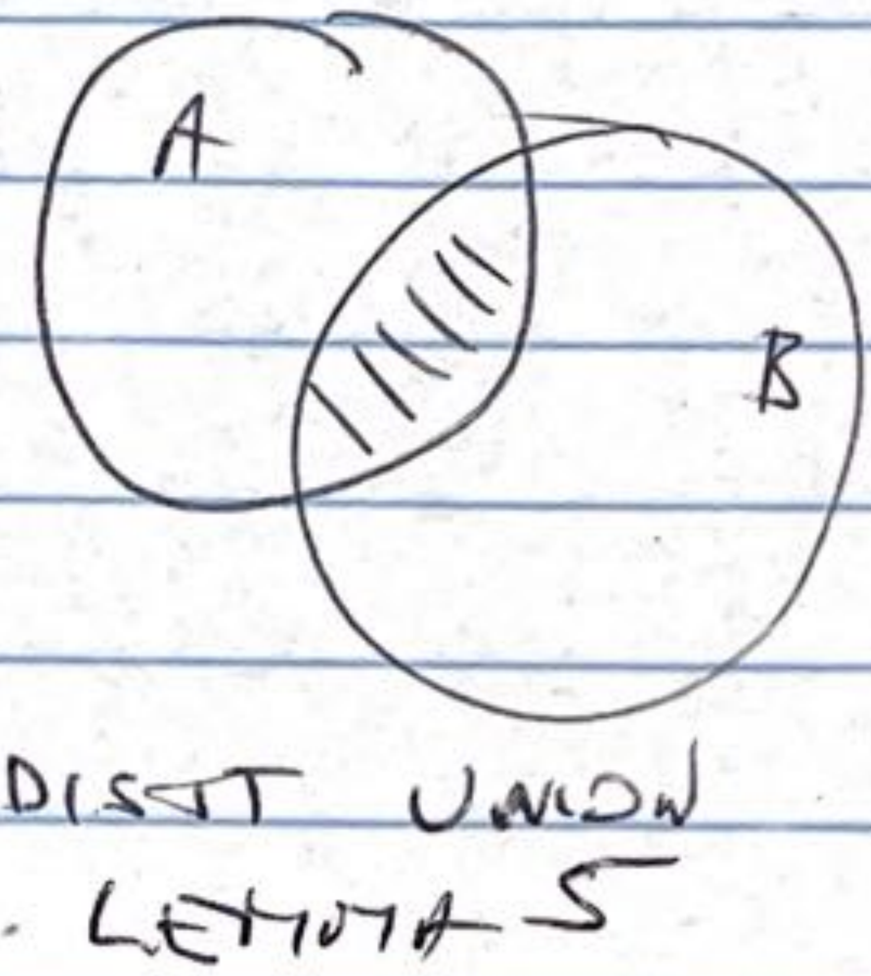
AND $\quad G \sim K \subseteq (G_1 \sim K_1) \cup (G_2 \sim K_2)$

So

$$\lambda(G \sim K) \leq \lambda(G_1 \sim K_1) + \lambda(G_2 \sim K_2) < \varepsilon$$

So $\quad A \sim B$ is Leb M'ble.

(2) $\boxed{A \cap B}$ $\quad A \cap B = A \sim (A \sim B)$



So apply (1) Twice

(3) $\boxed{A \cup B}$ $\quad A \cup B = (A \sim B) \cup B \quad$ DISJT UNION
So $A \cup B$ is Leb M'ble by LEMMA 5