THE CANTOR SET

[A, 2D], [J, 2A]

Countable sets have measure zero, eg $\lambda(\mathbb{Q}) = 0$.

Q $\exists$? Uncountable set with measure zero?

A YES!

The CANTOR SET, C, IS A SUBSET OF $[0,1]$:

1. C IS COMPACT + hence $\lambda(C)$ $\exists$
2. $\lambda(C) = 0$
3. C does not contain any intervals
4. C is UNCOUNTABLE.
5. C can be put into 1-1 correspondence with $\mathbb{R}$.

CONSTRUCTION

$$C = [0,1] \sim \bigcup_{n=1}^{\infty} G_n$$

where

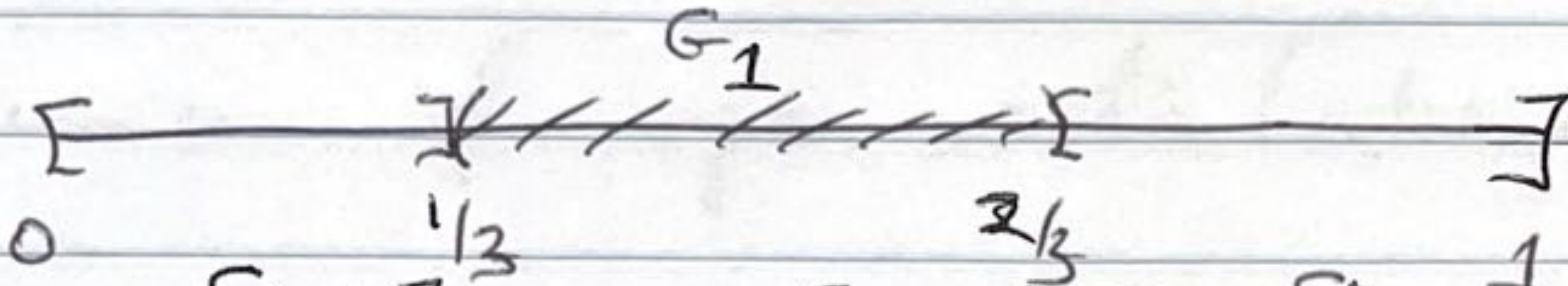
$$G_1 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

G_n is union of all middle-third open intervals in the intervals of

$$[0,1] \sim \bigcup_{j=1}^{n-1} G_j$$

<u>So</u> Start with $[0, 1]$,

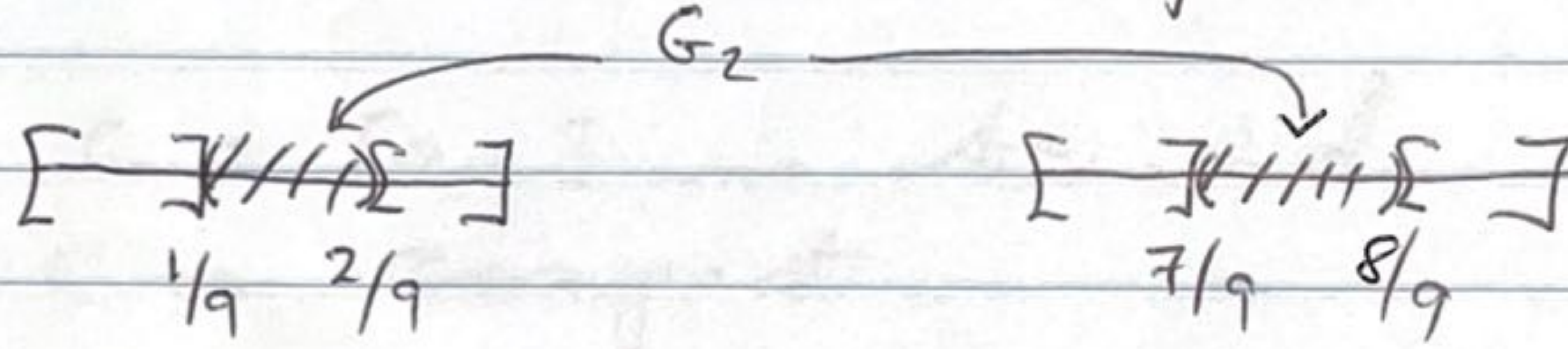
Remove middle-third open interval



P_1 $[0, 1] \sim G_1 = [0, \frac{1}{3}] \cup [2/3, 1]$

Now we have 2 closed intervals.

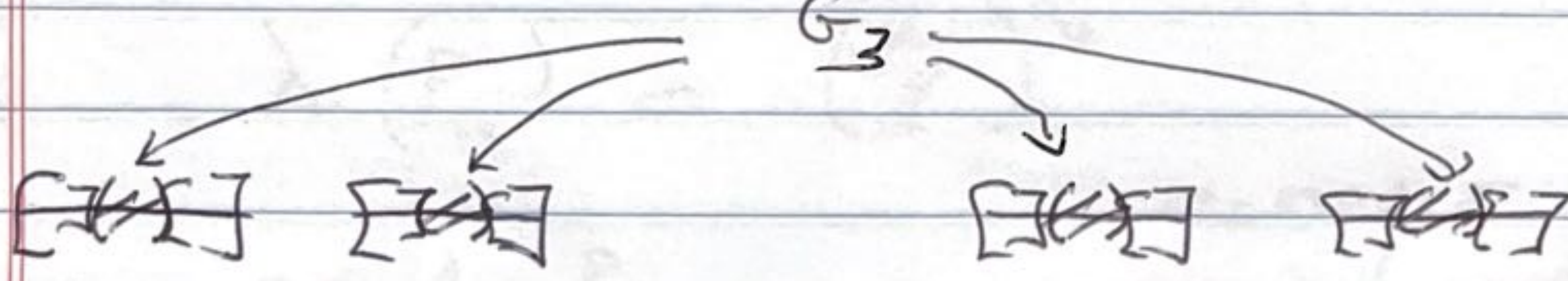
Remove middle-third open interval of each



Now we have 4 closed intervals

P_2 $[0, 1] \sim (G_1 \cup G_2) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

Remove middle-third open interval of each of these 4 intervals.



$P_3 = [0, 1] \sim (G_1 \cup G_2 \cup G_3) =$ 8 intervals removed

$[0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}]$

$\cup [2/3, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1]$

CLAIM I $\quad C$ is COMPACT.

PF $\quad C^c = (-\infty, 0) \cup \left(\bigcup_{k=1}^{\infty} G_n\right) \cup (1, \infty)$

is a union of open sets (intervals) and so is open.

So C is closed

But $C \subset [0,1]$ is bounded
So C is compact.

CLAIM II $\quad \lambda(C) = 0$

PF $\quad \forall N,\quad C$ is a subset of $P_N = [0,1] \sim \bigcup_{k=1}^{N} G_k$

which is a union of 2^N closed intervals of measure $\frac{1}{3^N}$

So $\quad \lambda(P_N) = \left(\frac{2}{3}\right)^N$

Since each P_N is ~~a special polygon~~ COMPACT

$$0 \le \lambda(C) \overset{C2}{\le} \lambda(P_N) = \left(\frac{2}{3}\right)^N \to 0 \text{ as } N \to \infty.$$

So $\lambda(C) = 0$. $\square$

CLAIM III

C does not contain any intervals consisting of more than 1 element

PF If $\exists\ I = [a,b] \subseteq C$ Then by (C2)

$$b - a = \lambda(I) \leq \lambda(C) = 0$$

So $b = a$ □

IDEA To determine whether or not $x \in C$ look at base 3 (ternary) expansion of x.

DECIMAL EXPANSIONS (BASE 10)

Let $x \in [0,1]$. Then $\exists \{a_k\}_{k=1}^{\infty}$ with each $a_k \in \{0, 1, 2, \dots, 8, 9\}$:

$$x = \sum_{k=1}^{\infty} \frac{a_k}{10^k} \quad (*)$$

Write $x = 0.a_1 a_2 a_3 \dots$

FACT The series expansion ⊛ <u>may</u> not be unique

EG $0.\underline{1} = 0.100\ldots$

$= 0.0999\ldots$

as $\sum_{k=2}^{\infty} \frac{9}{10^k} = \frac{9}{10} \sum_{k=1}^{\infty} \frac{1}{10^k} = \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}} - 1\right)$

$= 0.1$

TERNARY EXPANSIONS (BASE 3)

Let $x \in [0,1]$. Then $\exists \{a_k\}_{k=1}^{\infty}$ with each

$a_k \in \{0, 1, 2\}$ so that

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

WRITE $x = 0._3 a_1 a_2 a_3 \ldots$

Just as before, expansion may not be unique:

$\frac{1}{3} = 0._3 100\ldots = 0._3 0222\ldots$

LEMMA

Let $x \in [0,1]$.

① x has at most 2 base 3 expansions.

② If x has 2 base 3 expansions
Then one of them must terminate.

PF TEDIOUS

COR

The $x \in [0,1]$ which have 2 base 3 expansions are of the form

$$x = \frac{p}{3^n} \quad \text{for an integer } p.$$

PF

Terminating base 3 expansions are of form

$$x = \sum_{k=1}^{n} \frac{a_k}{3^k} = \frac{1}{3^n} \underbrace{\left(\sum_{k=1}^{n} a_k 3^{n-k} \right)}_{= p \in \mathbb{Z}_+}$$

□

RELATION TO CANTOR SET

CLAIM I

$x \in G_1 = \left(\frac{1}{3}, \frac{2}{3}\right) \iff$ Every base 3 expansion of x has form

$$x = 0{\cdot}_3 1 a_2 a_3 \dots$$

PF

$\boxed{\Rightarrow}$ Let $x \in (1/3, 2/3)$.

If $a_1 = 0$ then $x = 0{\cdot}_3 0 a_2 a_3 \dots$

$$\leq 0{\cdot}_3 022\dots = 1/3 \text{ X}$$

If $a_1 = 2$ then $x = 0{\cdot}_3 2 a_2 a_3 \dots$

$$= \frac{2}{3} + \frac{a_2}{3^2} + \dots \geq 2/3 \text{ X}$$

So $a_1 = 1$ holds

$\boxed{\Leftarrow}$ IF $a_1 = 1$ then

$x = 0{\cdot}_3 1 a_2 a_3 \dots \geq 0{\cdot}_3 1 = \frac{1}{3}$

$x = 0{\cdot}_3 1 a_2 a_3 \dots \leq 0{\cdot}_3 122 \dots = 0{\cdot}_3 2 = 2/3.$

So $x \in [\frac{1}{3}, 2/3]$.

However if $x = \frac{1}{3}$ or $x = \frac{2}{3}$ Then by CGR

x has 2 base 3 expansions, the 2nd being

$1/3 = 0_3\,022\ldots$

$2/3 = 0_3\,22\ldots$

which do not have $a_1 = 1$ ∎

CLAIM II

$x \in \bigcup_{j=1}^{\infty} G_j \iff$ Every base 3 expansion of x has at least one 1 in it.

PF (HWK)

NEGATING CLAIM II we get

THM The Cantor C is set of #s in $[0,1]$ That have (at least one) base 3 expansion containing only 0's and 2's.

EX 1. The endpts of each C_n are in C (as they are never ~~re~~removed)

2. $\frac{1}{4}, \frac{9}{13} \in C$ but are not endpts of a C_n.

PROP C IS UNCOUNTABLE. and can be put into 1-1 correspondence with $\mathbb{R}$

PF Define

$$F : C \longrightarrow [0,1] \text{ by}$$

$$F\left(\sum_{j=1}^{\infty} \frac{a_j}{3^j}\right) = \sum_{j=1}^{\infty} \frac{[a_j/2]}{2^j}$$

where we always choose our base 3 expansion to contain only 0s and 2s.

UEV F is 1-1 and ONTO.

So C is uncountable as $[0,1]$ is