LECTURE 14 LEBESGUE MEASURE ON EUCLIDEAN SPACE (CONT'D) [J, #2]

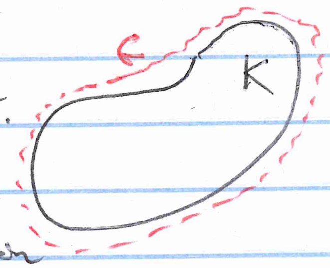
CONSTRUCTION OF LEBESGUE MEASURE

STAGE 4 COMPACT SETS IDEA: APPROXIMATE COMPACT SET FROM WITHOUT BY OPEN SETS

DEF 1 Let $K \subseteq \mathbb{R}^n$ be compact.
Define

$$\lambda(K) := \inf\{\lambda(G) \,/\, K \subseteq G,\ G \text{ open}\}$$

NOTICE: Any Special Polygon P is compact.
We need def^{ns} in STAGES 2, 4 TO AGREE.



LEMMA 2 Let P be a special polygon

Then

$$\lambda_{\underset{\uparrow\ \text{STAGE 2}}{\text{SPEC POLY}}}(P) = \lambda_{\underset{\uparrow\ \text{STAGE 4}}{\text{COMPACT}}}(P)$$

where

~~DEF~~ $$\lambda_{\text{SPEC POLY}}(P) = \sum_{k=1}^{N} \lambda(I_k)$$

with $$P = \bigcup_{k=1}^{N} I_k,$$ non-overlapping special rectangles.

NOTE
Stage 2, 4 def^{ns} are connected through Stage 3 (Open Sets)

~~Suppose $P \subseteq G$.~~

Recall If $G \subseteq \mathbb{R}^n$ is open Then

$$\lambda(G) = \sup\{\lambda_{\text{SPEC POLY}}(P) \,/\, P \subseteq G \text{ is special polygon}\}$$

PF

① $\lambda_{\text{SPEC POLY}}(P) \le \lambda_{\text{COMPACT}}(P)$

PF By defⁿ of λ_{COMPACT}:

LET $\varepsilon > 0$, $\exists$ open G_* with $P \subseteq G_*$ and

$$\lambda_{\text{COMPACT}}(P) + \varepsilon \ge \lambda(G_*).$$

Also by defⁿ of $\lambda(G_*)$ we have

$$\lambda(G_*) \ge \lambda_{\text{SPEC POLY}}(P)$$ as $P \subseteq G_*$ is a SPEC POLY.

So $\forall \varepsilon > 0$

$$\lambda_{\text{SPEC POLY}}(P) \le \lambda(G_*) \le \lambda_{\text{COMPACT}}(P) + \varepsilon$$

$$\therefore \quad \lambda_{\text{SPEC POLY}}(P) \le \lambda_{\text{COMPACT}}(P)$$

② $\lambda_{\text{COMPACT}}(P) \le \lambda_{\text{SPEC POLY}}(P)$.

LET

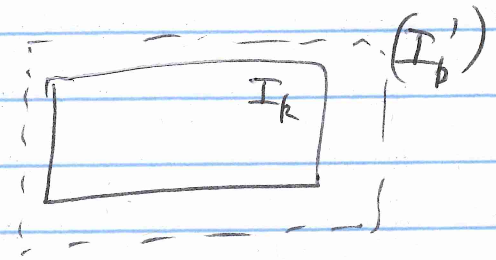
$P = \bigcup_{k=1}^{N} I_k$ be non-overlapping union of special rectangles.

Need to get G open with $P \subseteq G$.

For that let $\varepsilon > 0$.

Choose special rectangles I_k' so that

$$I_k \subseteq (I_k')^\circ$$



and $$\lambda(I_k') < \lambda(I_k) + \frac{\varepsilon}{N}$$

Let

$$G = \bigcup_{k=1}^{N} (I_k')^\circ$$

open. [NO LONGER NONOVERLAPPING]

and by construction $P \subseteq G$.

So by def^n of $\lambda_{\text{COMPACT}}(P)$

$$\begin{aligned}
\lambda_{\text{COMPACT}}(P) &\leq \lambda(G) \\
&\leq \sum_{k=1}^{N} \lambda\left((I_k')^\circ\right) && \text{PROP 5, (05)} \\
&= \sum_{k=1}^{N} \lambda(I_k') && \text{PROP 5 (07)} \\
&< \sum_{k=1}^{N} \left[\lambda(I_k) + \frac{\varepsilon}{N}\right] \\
&= \lambda_{\text{SPEC POLY}}(P) + \varepsilon
\end{aligned}$$

Since true $\forall \varepsilon > 0$: $\lambda_{\text{COMPACT}}(P) \leq \lambda_{\text{SPEC POLY}}(P)$ $\Box$

PROP 3 Let $K \subseteq \mathbb{R}^n$ be COMPACT. Then

(C1) $0 \leq \lambda(K) < \infty$

(C2) $K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$

(C3) $\lambda(\underbrace{K_1 \cup K_2}_{\text{COMPACT}}) \leq \lambda(K_1) + \lambda(K_2)$

(C4) If $K_1 \cap K_2 = \emptyset$ Then $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$

PF

(C1) (a) Since compact sets are bounded $\exists R > 0$:
$K \subset B_R(0) := G$

So $\lambda(K) \leq \lambda(B_R(0)) < \infty$.

(b) $\lambda(G) \geq 0 \ \forall G$ So $\lambda(K) = \inf\{\lambda(G) / K \subset G\} \geq 0$.

(C2) Let G be open with $K_2 \subseteq G$.

Then $K_1 \subseteq K_2 \subseteq G$.

So $\underset{LB}{\lambda(K_1)} \leq \{\lambda(G) \ / \ \forall G \text{ with } K_2 \subseteq G\}$

So $\lambda(K_1) \leq \underset{GLB}{\lambda(K_2)}$.

(C3) LET $\varepsilon > 0$

Choose $G_j \supseteq K_j$ so that $\underbrace{\underbrace{\lambda(K_j)}_{\text{GLB}} + \frac{\varepsilon}{2}}_{\text{NOT LB}} > \lambda(G_j)$, $j = 1, 2$

Then $K_1 \cup K_2 \subseteq \underbrace{G_1 \cup G_2}_{\text{OPEN}}$ holds

and so

$$\underset{\text{GLB}}{\lambda(K_1 \cup K_2)} \leq \lambda(G_1 \cup G_2)$$

$$\leq \lambda(G_1) + \lambda(G_2) \quad \text{PROPS (O5)}$$

$$< \left[\lambda(K_1) + \frac{\varepsilon}{2}\right] + \left[\lambda(K_2) + \varepsilon/2\right]$$

$$= \lambda(K_1) + \lambda(K_2) + \varepsilon.$$

Since true $\forall \varepsilon > 0$, $\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$

(C4) The proof relies on the following COR of Lecture 12, Lemma 11 (Lebesgue Covering Lemma)

COR 4

Let K be compact and F closed, $K \cap F = \emptyset$.

Then $\exists \varepsilon > 0$: $\forall x \in K$, $\forall y \in F$

$$d(x, y) \geq \varepsilon.$$

NB $\lambda(K_1) + \lambda(K_2) \leq \lambda(K_1 \cup K_2)$

IDEA Approximate $K_1 \cup K_2$ from without by open G and construct disjoint $G_1, G_2 \subseteq G$ open with $K_j \subseteq G_j$

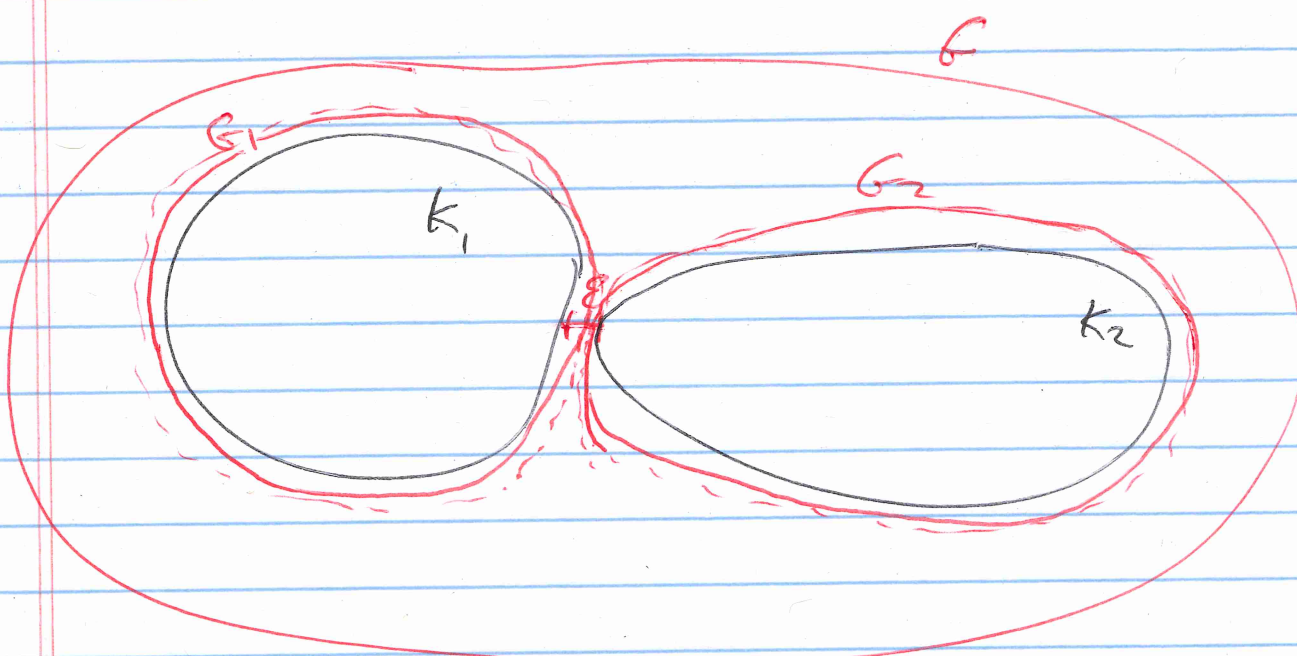
DETAILS Apply COR 4 with $K = K_1$, $F = K_2$ to get

$\exists \, \varepsilon > 0 : \forall x_1 \in K_1 \; \forall x_2 \in K_2 \quad d(x_1, x_2) \geq \varepsilon.$

Let G be open with $K_1 \cup K_2 \subseteq G$

Set $G_j = G \cap \left[\bigcup_{x \in K_j} B(x, \frac{\varepsilon}{2}) \right]$ open

So $K_j \subseteq G_j \subseteq G$, $\quad G_1 \cup G_2 \subseteq G$



CLAIM (UI✓) By COR 4 $\quad G_1 \cap G_2 = \emptyset.$

So $\lambda(K_1) + \lambda(K_2) \leq \lambda(G_1) + \lambda(G_2)$

INF FOR $\lambda(G_1)$

$\overset{\text{PROP 5}}{=} \lambda(G_1 \cup G_2)$

06

$\leq \lambda(G)$ as $G_1 \cup G_2 \subseteq G$

Finally we can choose G with $K_1 \cup K_2 \subseteq G$

so That $\lambda(G)$ is at most ε larger Than $\lambda(K_1 \cup K_2)$

So

$$\lambda(K_1) + \lambda(K_2) \leq \lambda(G) \leq \lambda(K_1 \cup K_2) + \varepsilon$$

So $\lambda(K_1) + \lambda(K_2) \leq \lambda(K_1 \cup K_2)$.

— o —

PROOF OF COR 4

Lebesgue Lemma If $K \subseteq \bigcup_{i \in I} G_i$ G_i open

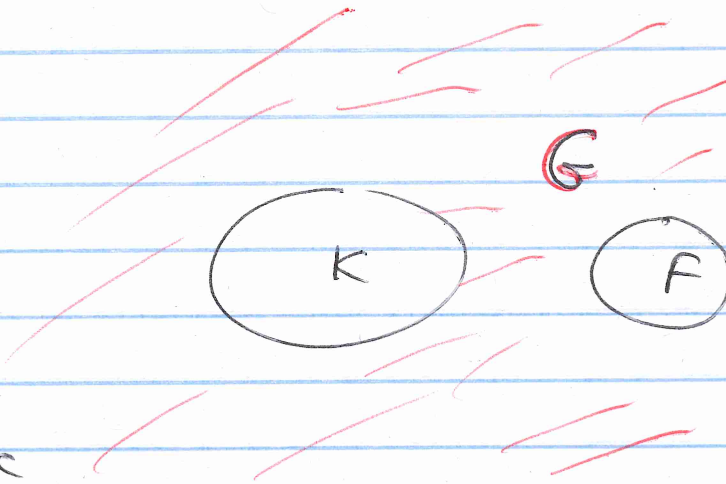
Then $\exists \varepsilon > 0 : \forall x \in K \ \exists i \in I : B(x, \varepsilon) \subseteq G_i$

Apply to

$K \subseteq G$ (A single open set) to get

$\exists \varepsilon > 0 : \forall x \in K \ B(x, \varepsilon) \subseteq G$.

Then to prove Cor 4, we have K compact, F closed $K \cap F = \emptyset$.



Let $G = F^c$.

So $K \subseteq G$

So $\exists \varepsilon > 0 : \forall x \in K$

$$B(x, \varepsilon) \subseteq G = F^c$$

So $\forall y \in F$, $y \notin B(x, \varepsilon)$

So $\forall x \in K, \forall y \in F$ $d(x,y) \geq \varepsilon$ ✓✓

$\square$

NOTE By HWK we know this result doesn't hold ~~for K, F~~ two closed sets. At least one of them must be compact.