PROPN 5 Let G, G_k be an open subsets of $\mathbb{R}^n$.

O1 $0 \leq \lambda(G) \leq \infty$.

O2 $\lambda(G) = 0 \iff G = \emptyset$

O3 $\lambda(\mathbb{R}^n) = \infty$.

O4 $G_1 \subseteq G_2 \implies \lambda(G_1) \leq \lambda(G_2)$

O5
$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

O6 IF the G_k are DISJOINT [ie $G_k \cap G_l = \emptyset$ for $k \neq l$]

THEN
$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) = \sum_{k=1}^{\infty} \lambda(G_k)$$

O7 If P is a special polygon then P° is an open set and

$$\lambda(P^\circ) = \lambda(P)$$

ie Our defns of λ for Special Polygons and Open Sets agree when they should

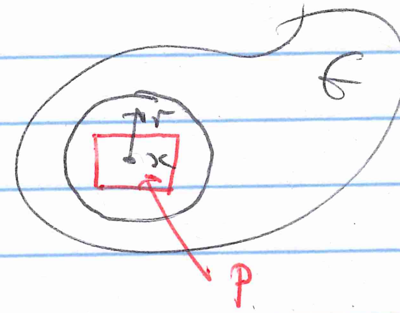
Pf

(02) $\boxed{\Leftarrow}$ By def'n if $G = \emptyset$ then $\lambda(G) = 0$

$\boxed{\Rightarrow}$ Show $G \neq \emptyset \Rightarrow \lambda(G) > 0$.

Well since G is open $\exists$ special polygon P with $P_* \subseteq G$ and $\lambda(P_*) > 0$

So



$$\lambda(G) = \sup_P \lambda(P) \geq \lambda(P_*) > 0$$

(03) $\forall r \in \mathbb{R}^+$, $P_r = I_r = [-r, r]^n \subseteq \mathbb{R}^n$

Also $\lambda(P_r) = (2r)^n \longrightarrow \infty$ as $r \longrightarrow \infty$.

So $\Lambda = \{ \lambda(P) / P \subseteq \mathbb{R}^n, P \text{ special polygon} \}$

is an unbounded set. So $\lambda(\mathbb{R}^n) = \infty$.

(04) Let $G_1 \subseteq G_2$

$$\Lambda_{G_j} = \{ \lambda(P) / P \subset G_j, P \text{ special polygon} \}$$

~~Then~~ Now if $P \subset G_1$ and $G_1 \subseteq G_2$ then $P \subseteq G_2$

So $\Lambda_{G_1} \subset \Lambda_{G_2} \subseteq \mathbb{R}$

So

$$\lambda(G_2) = \sup(\Lambda_{G_2})$$

is an upper bound for Λ_{G_1}.

So $\lambda(G_1) \le \lambda(G_2)$

LUB UB

05 NOTE G_k open $\forall k \Rightarrow \bigcup_{k=1}^{\infty} G_k$ is open too.

So $\lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$ is defined.

We know

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) = \sup_P \left\{ \lambda(P) \,/\, P \subseteq \bigcup_{k=1}^{\infty} G_k \right\}$$

Let P be <u>any</u> special polygon for which

$$P \subseteq \bigcup_{k=1}^{\infty} G_k .$$

CLAIM

$$\lambda(P) \le \underbrace{\sum_{k=1}^{\infty} \lambda(G_k)}_{\text{UPPER BOUND FOR THE } \lambda(P)\text{'s}} .$$

~~HENCE~~ $\cancel{\lambda\left(\bigcup_{k=1}^{\infty} G_k\right)} \le$

BY CLAIM
$$\lambda\Big(\underbrace{\bigcup_{k=1}^{\infty} G_k}_{\text{LEAST UB FOR } \lambda(P)!}\Big) \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

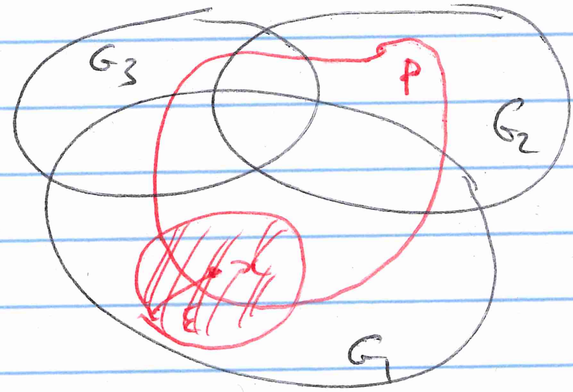
PF OF CLAIM

Since P is a special polygon, P is COMPACT

and $P \subseteq \bigcup_{k=1}^{\infty} G_k$ gives an OPEN COVER for P.

By Lecture 12, Lemma 11 (Lebesgue # Lemma)

$\exists\, \varepsilon > 0 : \forall x \in P \;\; \exists k : \quad B(x, \varepsilon) \subset G_k$



Now since P is a special polygon

$$P = \bigcup_{j=1}^{N} I_j$$

where I_j are special rectangles, that are NON-OVERLAPPING.

ASIDE (VERY)

1. LET $A \subseteq \mathbb{R}^n$. The DIAMETER of A is

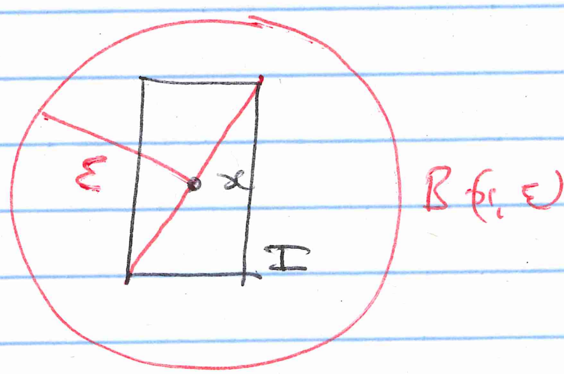
$$\operatorname{diam}(A) = \sup \{ d(x,y) \mid x, y \in A \}$$

2. IF $A_1 \subseteq A_2$ THEN $\operatorname{diam}(A_1) \leq \operatorname{diam}(A_2)$.

3. $\operatorname{diam} B(x, r) = 2r$.

4. IF $\operatorname{diam}(I) < 2\varepsilon$ where I is a special rectangle THEN

$$I \subset B(x, \varepsilon) \quad \text{where } x \text{ is the center of } I.$$



By possibly subdividing each of rectangles I_j

in $$P = \bigcup_{j=1}^{N} I_j$$

where can assume $\operatorname{diam}(I_j) \leq 2\varepsilon$

where ε = Lebesgue # of Covering $P \subset \bigcup_{k=1}^{\infty} G_k$

GIVEN I_j, let x_j = Center of I_j

Then by ④ above and Lebesgue Lemma

$$I_j \subset B(x_j, \varepsilon) \subset G_k \text{ for some } k.$$

SUMMARY

$$P = \bigcup_{j=1}^{N} I_j, \quad I_j \text{ are special rectangles}$$

with each I_j entirely contained in

at least one of G_k's.

GOAL ASSIGN SOME OF I_j's TO G_1, SOME TO G_2, ETC.

Since $\lambda(P) = \sum \lambda(I_j)$ and since $\quad \lambda(\text{All the } I_j\text{'s assigned to } G_k) \leq \lambda(G_k)$

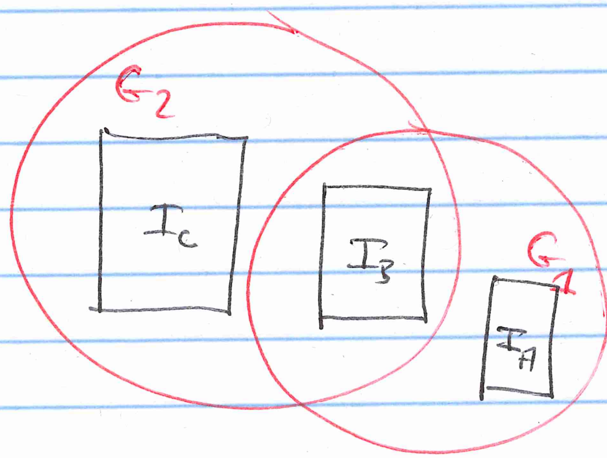
we will get

$$\lambda(P) \leq \sum_{k=1}^{\infty} \lambda(G_k).$$

RIGOROUS DETAILS

Let $\quad P_1 = \bigcup \{ I_j \,/\, I_j \subset G_1 \} \subset G_1$

$\qquad P_2 = \bigcup \{ I_j \,/\, I_j \subset G_2, \text{ but } I_j \not\subset G_1 \} \subset G_2$



$I_A, I_B \subset G_1.$

$P_1 = I_A \cup I_B \cup \ldots$

$\boxed{P_1 \subset G_1}$

$I_B, I_C \subset G_2$

But $\quad I_B \subset G_1$

So $\quad P_2 = I_C \cup \ldots$

$\boxed{P_2 \subset G_2}$

$$P_k = \bigcup \{ I_j \,/\, I_j \subset G_k \text{ BUT } I_j \not\subset \bigcup_{l=1}^{k-1} G_l \} \subset G_k$$

By construction

① $P = \bigcup_{k=1}^{\infty} P_k$ and P_k's are NON_OVERLAPPING SPECIAL POLYGONS

② Since each I_j belongs to just one of P_k's

and $\exists N < \infty$ of I_j's

AT MOST N of P_k's are NON-EMPTY.

③ $P_k \subset G_k \Rightarrow \lambda(P_k) \leq \lambda(G_k)$ by def^n $\lambda(G_k)$

$\therefore$

$$\lambda(P) = \underbrace{\sum_k \lambda(P_k)}_{\text{FINITE SUM}} \qquad \text{by P2 of PROP 3}$$

$$\leq \sum_{k=1}^{\infty} \lambda(G_k) \qquad \text{as required to prove CLAIM}$$

$\square$

06 IF $\{G_k\}_{k=1}^{\infty}$ are disjt open sets THEN

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) = \sum_{k=1}^{\infty} \lambda(G_k).$$

PF

By (∞) NTS $\boxed{\sum_{k=1}^{\infty} \lambda(G_k) \le \lambda\left(\bigcup_{k=1}^{\infty} G_k\right)}$

By def^n of $\lambda(G_k) = \sup_{P_k} \{\lambda(P_k) / P_k \subset G_k\}$

Let $N < \infty$

LET $\varepsilon > 0$. $\exists\ P_k \subset G_k$:

$$\lambda(G_k) < \lambda(P_k) + \varepsilon/N$$

So

$$\sum_{k=1}^{N} \lambda(G_k) < \sum_{k=1}^{N} \lambda(P_k) + \varepsilon$$

$$= \lambda\left(\bigcup_{k=1}^{N} P_k\right) + \varepsilon \qquad (1)$$

FINITE DISJT UNION OF SPECIAL POLYGONS (as G_k are disjt)

Now

$$\underbrace{\bigcup_{k=1}^{N} P_k}_{\text{A SPECIAL POLYGON!!}} \subseteq \bigcup_{k=1}^{N} G_k \subseteq \underbrace{\bigcup_{k=1}^{\infty} G_k}_{\text{OPEN SET.}}$$

LUB FOR

So
$$\lambda\left(\bigcup_{k=1}^{N} P_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \quad (2)$$

by defn of $\lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$.

So by (1), (2) and fact $\varepsilon > 0$ is arbitrary

$$\sum_{k=1}^{N} \lambda(G_k) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \quad (3)$$

~~Since (3) is~~

NOW $\quad S_N = \sum_{k=1}^{N} \lambda(G_k)$

is an increasing sequence of real #s

$\lambda(G_k) \geq 0 \ \forall k$

that by (3) is bounded above. So this sequence converges and

$$\sum_{k=1}^{\infty} \lambda(G_k) = \lim_{N\to\infty} S_N = \underbrace{\sup_N S_N}_{\text{LUB}} \overset{(3)}{\leq} \underbrace{\lambda\left(\bigcup_{k=1}^{\infty} G_k\right)}_{\text{UB.}}$$

$\square$

07 If P is a special polygon Then $\lambda(P^\circ) = \lambda(P)$

Recall $G = P^\circ$ = Interior of P is an open set.

PF

CLAIM I $\quad \lambda(P^\circ) \leq \lambda(P)$

PF

$$\lambda(P^\circ) = \sup_{Q} \{ \lambda(Q) \mid Q \subset P^\circ, \ Q \text{ special polygon} \}$$

If $Q \leq P^\circ$ Then $Q \subseteq P^\circ \subseteq P$

BOTH SPECIAL POLYGONS

So by PROP 3 P2 we have

$$\lambda(Q) \leq \underbrace{\lambda(P)}_{\text{UPPER BOUND FOR } \lambda(Q)\text{'s}}$$

So

$$\underset{\text{LUB}}{\lambda(P^\circ)} \leq \lambda(P)$$

$\square$

CLAIM II Let I be a Special Rectangle.

Then
$$\lambda(I^\circ) \geq \lambda(I)$$

PF Let $\varepsilon > 0$. $\exists$ Special Rectangle J_* slightly smaller than I in that

$$J_* \subset I^\circ, \quad \lambda(J_*) > \lambda(I) - \varepsilon$$

So by def^n $\lambda(I^\circ) = \sup_J \{ \lambda(J) \,/\, J \subset I \}$

we have

$$\lambda(I^\circ) > \lambda(J_*) > \lambda(I) - \varepsilon.$$

So since this is true $\forall \varepsilon > 0$,

$$\lambda(I^\circ) \geq \lambda(I) \qquad \square$$

CLAIM III

$$\lambda(P^\circ) \geq \lambda(P)$$

PF Express $P = \bigcup_{k=1}^{N} I_k$ Non Overlapping.

Then $\bigcup_{k=1}^{N} I_k^\circ \subseteq P^\circ$ (UE✓)

DISJT UNION.

So by 06

$$\lambda(P^\circ) \overset{\text{DEF SUP}}{\geq} \lambda\left(\bigcup_{k=1}^{N} I_k^\circ\right) \overset{06}{=} \sum_{k=1}^{N} \lambda(I_k^\circ)$$

$$\overset{\text{CLAIM II}}{\geq} \sum_{k=1}^{N} \lambda(I_k) \underset{\lambda(P)}{\overset{\text{DEF}}{=}} \lambda(P)$$

$\square$