LECTURE 13 — LEBESGUE MEASURE ON $\mathbb{R}^n$ — [J, #2]

GOAL: For a large class of subsets $A \subseteq \mathbb{R}^n$ assign a number $\lambda(A) \in [0, \infty]$ which "measures the size of A".

$n=1$ LENGTH

$n=2$ AREA

$n=3$ VOLUME

$\lambda(A) =$ LEBESGUE MEASURE OF SET A.

The sets A for which $\lambda(A)$ $\exists$ will be called "measurable".

We construct Lebesgue Measure in 6 STEPS.

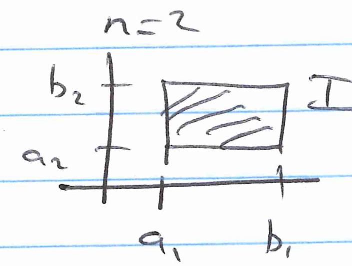
STAGE 0 — EMPTY SET: $\lambda(\emptyset) = 0$.

STAGE 1 — SPECIAL RECTANGLES

A SPECIAL RECTANGLE is a product of closed intervals:

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

$$= \{ x \in \mathbb{R}^n \,/\, a_i \le x_i \le b_i \text{ for } i = 1, \dots, n \}$$

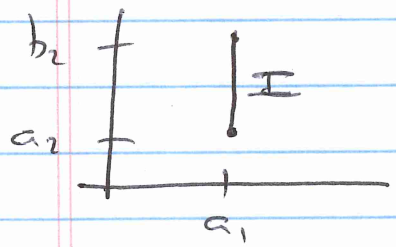


DEFINE $\lambda(I) = (b_1 - a_1) \cdots (b_n - a_n)$

"SIDES" MUST BE ALIGNED WITH COORD AXES

NOTE

1

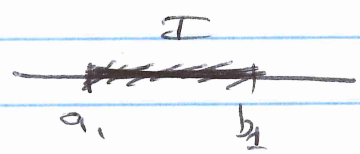


$I = [a_1, a_1] \times [a_2, b_2]$

is a line segment in $\mathbb{R}^2$.

$\lambda(I) = (a_1 - a_1)(b_2 - a_2) = 0$.

2 $I = [a_1, b_1]$ is a line segment in $\mathbb{R}$



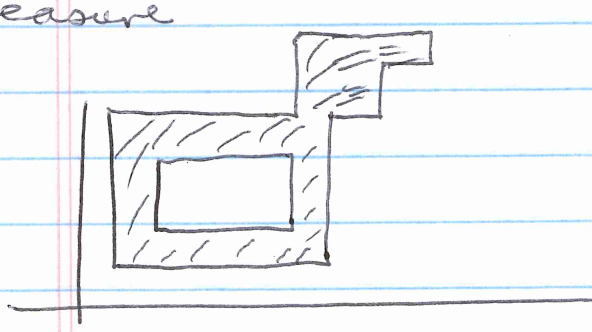
Here $\lambda(I) = b_1 - a_1 \neq 0$.

So we don't just care about set I but also about what $\mathbb{R}^n$ it is considered to be a subset of.

STAGE 2 SPECIAL POLYGONS

A SPECIAL POLYGON is a finite union of special rectangles, each of which has non-zero measure

EX



NOTE Special Rectangles are closed + bounded subsets of $\mathbb{R}^n$. So they are compact.

Since finite unions of compact sets are compact, Special Polygons are compact too.

DEF 1 Let P be a special polygon.

Decompose P as a finite union of NON-OVERLAPPING special rectangles:

$$P = \bigcup_{k=1}^{N} I_k$$

where $\quad I_k^\circ \cap I_\ell^\circ = \phi \quad$ for $k \neq \ell$

(Non-overlapping Interiors)

Define

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$$

LEMMA 2 $\lambda(P)$ is well defined in that

1. $\exists$ Decomp^n of P into a finite union of NON-OVERLAPPING special rectangles.

2. If $P = \bigcup_{k=1}^{N} I_k$ and $P^* = \bigcup_{k=1}^{N'} I_k'$ are two such decompositions then

$$\sum_{k=1}^{N} \lambda(I_k) = \sum_{k=1}^{N'} \lambda(I_k').$$

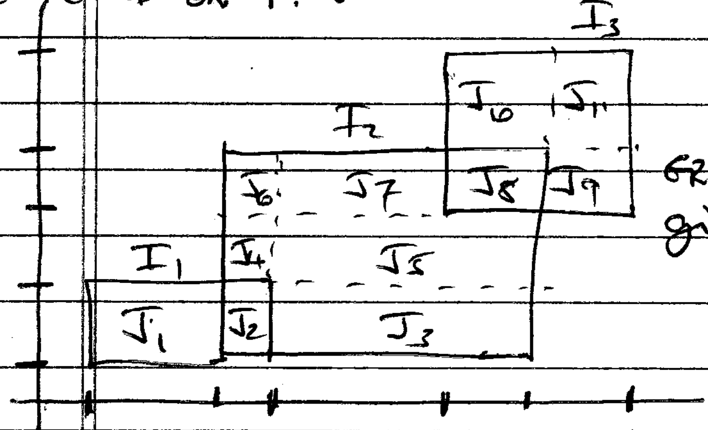
give the same value for $\lambda(P)$

PROOF IS TEDIOUS. WE WILL GIVE AN OUTLINE OF IDEAS.

PROOF OF 1

We know P is a union of possibly overlapping special rectangles. USE END PTS of these SRs to define a GRID on P.

EX



$P = I_1 \cup I_2 \cup I_3$. Overlapping.

GRID gives $P = J_1 \cup \dots \cup J_{11}$ Nonoverlapping

(2) Follows by setting $P' = P$ in the

<u>CLAIM</u>

Suppose $P \subset P'$ and

$$P = \bigcup_{k=1}^{N} I_k \quad , \quad P' = \bigcup_{k=1}^{N'} I_k'$$

express P, P' as union of non-overlapping special rectangles.

Then

$$\lambda(P) \leq \lambda(P')$$

<u>PF</u>

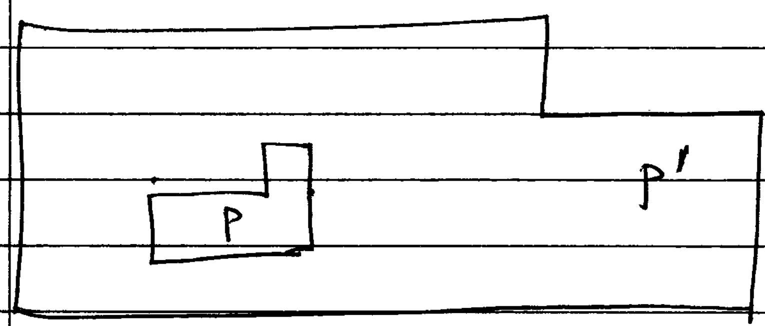
$$P' = P \cup P' = \bigcup_{k=1}^{N} I_k \cup \bigcup_{k=1}^{N'} I_k' \quad (*)$$

expresses P' as a union of (overlapping) special rectangles.

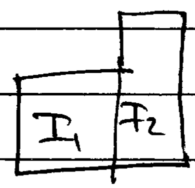
Use grid idea in (1) to ~~express~~ obtain a collection of non overlapping special rectangles based on the decomposition of P' in $(*)$.

So $\quad P' = J_1 \cup \dots \cup J_M \quad , \quad J_k$ non overlapping

EX

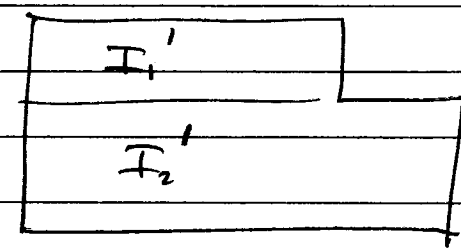


with



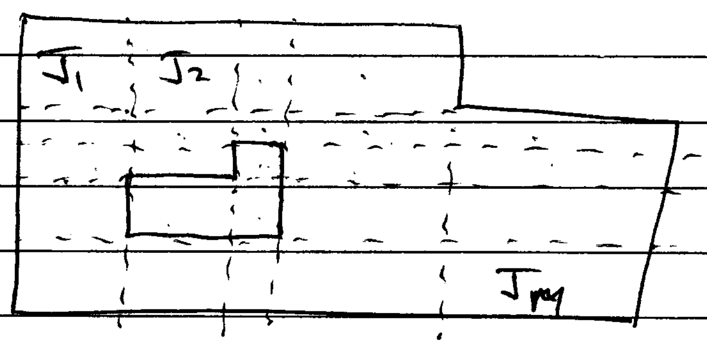
P

and



P'

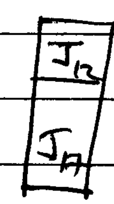
So get



J_l's NON OVERLAPPING

Clearly each I_k ~~for~~ P is a union of J_l's

and $$\lambda(I_k) = \sum_{J_l \subset I_k} \lambda(J_l)$$



I_2

Similary each I_k' for P' is a union of J_l's too and

$$\lambda(I_k') = \sum_{J_l \subset I_k'} \lambda(J_l)$$

Since all I_k's contributing to the sum

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$$

also contribute to the sum

$$\lambda(P') = \sum_{k=1}^{N'} \lambda(I_k')$$

it follows that
So $\lambda(P) \leq \lambda(P')$ must hold.

PROPN 3 IF P_1, P_2 are Special Polygons Then

P1 $P_1 \subset P_2 \implies \lambda(P_1) \leq \lambda(P_2)$

P2 IF P_1, P_2 are Non Overlapping Then

$$\lambda(P_1 \cup P_2) = \lambda(P_1) \cup \lambda(P_2)$$

ASIDE The union of 2 special polygons is another special polygon !!

PF **P1** This is just CLAIM in Pf of Lemma 2.

P2 Write $P_1 = \bigcup_{k=1}^{N} I_k^{(1)}$, $P_2 = \bigcup_{k=1}^{M} I_k^{(2)}$

as union of non overlapping special rectangles.

Then

$$P_1 \cup P_2 = \bigcup_{k=1}^{N} I_k^{(1)} \cup \bigcup_{k=1}^{M} I_k^{(2)}$$

~~is~~ ~~also~~ expresses $P_1 \cup P_2$ as union of nonoverlapping rectangles.

So

$$\lambda(P_1 \cup P_2) = \sum_{k=1}^{N} \lambda(I_k^{(1)}) + \sum_{k=1}^{M} \lambda(I_k^{(2)})$$

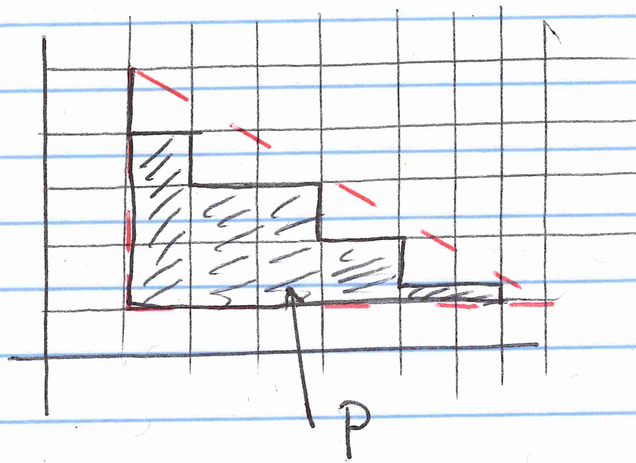
$$= \lambda(P_1) + \lambda(P_2) \qquad \square$$

STAGE 3 OPEN SETS.

IDEA

- APPROXIMATE an open set from within by special polygons.

EX G = INTERIOR OF TRIANGLE



$P \subset G$
SPECIAL POLYGON
OPEN

- THIS IDEA WORKS because $\forall x \in G \ \exists r > 0$: $B(x, r) \subset G$.

DEF 4 Let $G \subseteq \mathbb{R}^n$ be OPEN, $G \neq \phi$.

Define

$$\lambda(G) = \sup \{ \lambda(P) \mid P \subset G, \ P \text{ special polygon} \}$$

JUSTIFICATION FOR DEF

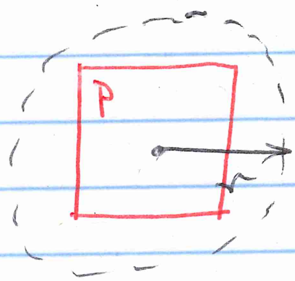
Let

$$\Lambda = \{ \lambda(P) \mid P \subseteq G, \ P \text{ special polygon} \} \subseteq \mathbb{R}$$

CLAIM $\Lambda \neq \emptyset$.

PF Let $x \in G$. $\exists\, B(x,r) \subseteq G$ where $r > 0$

$\exists$ Special Polygon $P \subset B(x,r)$.

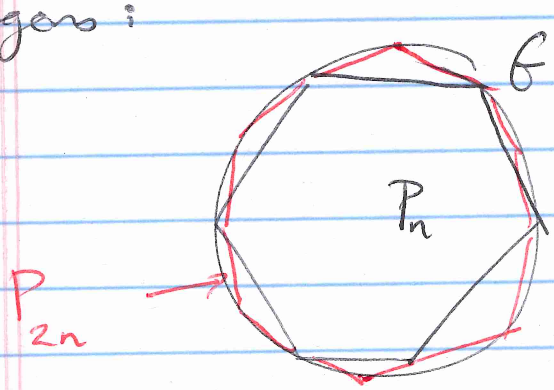


$\square$

IF Λ is bounded above Then $\lambda(G) = \sup \Lambda < \infty$ }

IF Λ is not bounded above Then set $\lambda(G) = \infty$.

NOTE THIS IDEA is similar to idea Greek's used to find area of circle by INSCRIBING regular polygons:



P_n = Regular n-gon

$$\lambda(G) = \lim_{n \to \infty} \lambda(P_n)$$

$$= \sup \lambda(P_n)$$