WHY DO WE NEED LEBESGUE?

[A, 1B]
~~[A, 1A]~~
[A, 2B]

I RIEMANN INTEGRAL IS NOT GOOD ENOUGH.

A R.I. DOES NOT HANDLE functions with many discontinuities.

EX1 DIRICHLET FUNCTION $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

For ANY $[a,b] \subseteq [0,1]$

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} f = 1$$

as $\mathbb{Q}$ is dense in $\mathbb{R}$.

So $\forall P$ $\quad L(f, P, [0,1]) = 0$
$\quad U(f, P, [0,1]) = 1$

So $\quad L(f,P) = 0, < 1 = U(f,P)$

So f NOT RI.

- On other hand $\mathbb{Q}$ is countable and $[0,1] \setminus \mathbb{Q}$ is NOT.
- W.R. Lebesgue Integral : $\int_0^1 f \, d\lambda = 1$ holds.

B R.I. DOES NOT WORK FOR UNBOUNDED FUNCTIONS

EX 2 $f : [0,1] \to \mathbb{R}$ is

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{\sqrt{x}} & 0 < x \le 1 \end{cases}$$

For any Partition P $\sup_{[x_0, x_1]} f = \infty$.

So would get $U(f, P, [0,1]) = \infty \quad \forall P$.

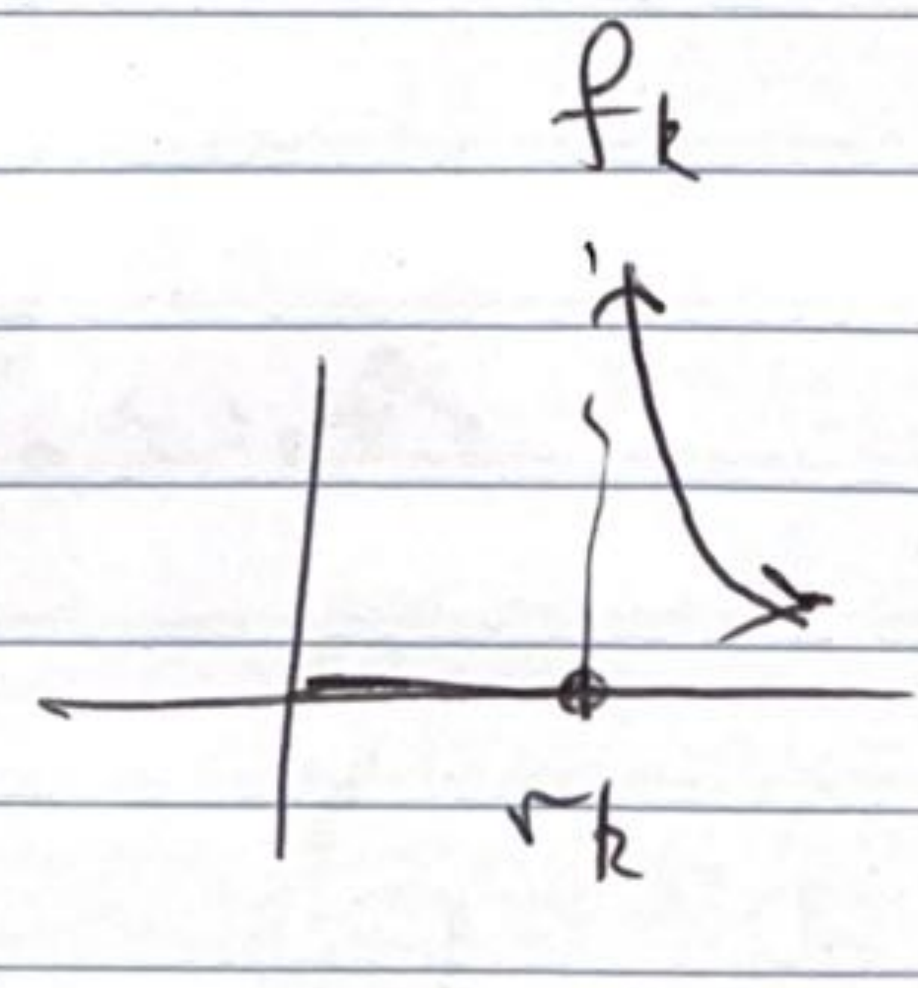
BUT Area under graph should be

$$A = \lim_{a \downarrow 0} \int_a^1 \frac{1}{\sqrt{x}}\, dx = 2. \quad \text{IMPROPER INTEGRAL}$$

EX 3 This example shows the improper integral trick doesn't always work

Let $\{r_k\}_{k=1}^{\infty}$ enumerate $\mathbb{Q} \cap (0,1)$.
Define

$$f_k(x) = \begin{cases} 0 & x \le r_k \\ \frac{1}{\sqrt{x - r_k}} & x > r_k \end{cases}$$



and $f : [0,1] \to [0,\infty]$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k(x)$$

NOTE f is unbounded on every subinterval of $[0,1]$. So can't split $[0,1]$ into subintervals + do an improper integral on each

BUT EXPECT

$$\text{Area Under } f = \int_0^1 f(x)\,dx$$

$$= \int_0^1 \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} f_k(x)\,dx$$

$$\overset{?}{=} \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} \frac{1}{2^k} f_k(x)\,dx$$

? WORKS IN LEBESGUE THEORY

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} \int_0^1 f_k(x)\,dx$$

$$\le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} \cdot 2 \qquad \text{by EX 1}$$

$$\le 1 + \frac{1}{2} + \frac{1}{4} + \dots \le 2 < \infty$$

(c) RI DOES NOT WORK WELL WITH LIMITS

Let $\{r_k\}_{k=1}^{\infty}$ enumerate $\mathbb{Q} \cap [0,1]$.
Set

$$f_k(x) = \begin{cases} 1 & x \in \{r_1, \dots r_k\} \\ 0 & \text{otherwise} \end{cases}$$ BOUNDED BY 1

UB✓

f_k is RI and $\int_0^1 f_k \, dx = 0$.

Let f be Dirichlet f^n

Then

$$f_k(x) \to f(x) \quad \forall x \in [0,1] \text{ as } k \to \infty.$$

BUT f is NOT R.I.

BOUNDED CONVERGENCE THM for RI

Suppose f_n is sequence of RI f^{ns} on $[a,b]$
and $\exists M > 0$:

$$|f_n(x)| \leq M \quad \forall x, n.$$

Suppose $f(x) = \lim_{n\to\infty} f_n(x)$ $\exists$ + is RI

Then

$$\int f \, dx = \lim_{n\to\infty} \int f_n \, dx$$

UNDESIRABLE ASSUMP^n

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II YOU CAN'T FIND LENGTH FOR EVERY SUBSET OF $\mathbb{R}$

THM $\nexists$ $\lambda : \{\text{SUBSETS OF } \mathbb{R}\} \to [0, \infty]$ so that

(a) $\lambda(I) = \text{LENGTH}(I)$ for every open interval $I \subset \mathbb{R}$

(b) If $\{A_k\}_{k=1}^{\infty}$ is disjoint coll^n of sets then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k)$$

WHOLE IS SUM OF PARTS

(c) $\forall A \subseteq \mathbb{R},\ t \in \mathbb{R}$

$$\lambda(t + A) = \lambda(A)$$

TRANSLATION INVARIANCE

NOTE This presents a challenge for def^n of Lebesgue Integral since

$\int f \, d\lambda$ is defined in terms of

Lebesgue measure which should have properties (a) – (c)

So we must identify a class of measurable subsets of $\mathbb{R}$ for which (a) – (c) hold.

PF $\mu(\emptyset) = 0$ as $\emptyset$ no open interval $\bar{c}$ length 0.

CLAIM I If $A \subseteq B$ Then $\lambda(A) \leq \lambda(B)$

PF $B = A \cup (B \sim A)$ DISJT UNION

So by (1)

$$\lambda(B) = \lambda(A) + \lambda(B \sim A) \geq \lambda(A)$$

CLAIM II $\lambda([a,b]) = b - a$

PF $\forall \varepsilon > 0$

$$(a,b) \subseteq [a,b] \subseteq (a-\varepsilon, b+\varepsilon)$$

$$\text{So } b-a = \lambda((a,b)) \overset{\mathrm{I}}{\leq} \lambda([a,b]) \overset{\mathrm{I}}{\leq} \lambda(a-\varepsilon, b+\varepsilon) = b-a+2\varepsilon.$$

So $\lambda((a,b)) = b-a$. D.

CLAIM III

For any sequence of subsets $A_1, A_2, \ldots$ of $\mathbb{R}$

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

PF $\bigcup_{k=1}^{\infty} A_k = A_1 \cup (A_2 \sim A_1) \cup A_3 \sim (A_1 \cup A_2) \cup \ldots$

DISJT UNION

So $\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda(A_1) + \lambda(A_2 \sim A_1) + \lambda(A_3 \sim (A_1 \cup A_2)) + \dots$

$$\overset{\mathrm{I}}{\leq} \lambda(A_1) + \lambda(A_2) + \lambda(A_3) + \dots$$

$$= \sum_{k=1}^{\infty} \lambda(A_k) \qquad \square$$

Given these preliminaries we obtain $\times$ if (a) – (c) hold:

For each $a \in [-1,1]$ let

$$\tilde{a} = \{ c \in [-1,1] \mid a - c \in \mathbb{Q} \}$$

= EQUIVALENCE CLASS OF a under $a \sim c$ if $a - c \in \mathbb{Q}$.

By disjointness property of equivalence classes we have

__CLAIM IV__ Either $\tilde{a} \cap \tilde{b} = \phi$ or $\tilde{b} = \tilde{a}$

and

__CLAIM V__

$$[-1,1] = \bigcup_{a \in [-1,1]} \tilde{a}$$

By AXIOM OF CHOICE $\exists$ SET V containing ! elt of each of sets $\tilde{a}$ for $a \in [-1,1]$.

LET $[-2,2] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$

CLAIM VI

$$[-1,1] \subseteq \bigcup_{k=1}^{\infty} (r_k + V)$$

PF

Let $a \in [-1,1]$ and let $v \in V$ be ! elt of $\tilde{a}$ in V

Then $a - v \in \mathbb{Q}$

Now $a, v \in [-1,1] \Rightarrow a - v \in [-2,2]$

So $a - v = r_k$ for some k

$a \in r_k + V$. $\square$

CLAIM VII

$$\lambda(V) > 0$$

PF $2 \overset{\text{II}}{=} \lambda([-1,1]) \overset{\text{III}}{\leq} \sum_{k=1}^{\infty} \lambda(r_k + V) \overset{\text{©}}{=} \sum_{k=1}^{\infty} \lambda(V)$

So $\lambda(V) \neq 0$.

CLAIM VIII

$$(r_k + V) \cap (r_\ell + V) = \phi \quad \text{for } k \neq \ell$$

PF

If $t \in (r_k + V) \cap (r_\ell + V)$

Then

$$t = r_k + v_1 = r_\ell + v_2 \qquad \text{for } v_1, v_2 \in V$$

$$v_1 - v_2 = r_\ell - r_k \in \mathbb{Q} = \tilde{0}$$

So $v_1 = v_2$ by construction of V

So $r_\ell = r_k$.

So $\ell = k$ ∎

FOR EACH n:

$$\bigcup_{k=1}^{n} (r_k + V) \overset{*}{\subseteq} [-3, 3]$$

as $r_k \in [-2, 2]$, $V \subseteq [-1, 1]$

So by Claim VIII

$$\sum_{k=1}^{n} \lambda(r_k + V) = \lambda\left(\bigcup_{k=1}^{n} (r_k + V)\right) \overset{*}{\leq} 6$$

But by (c)

$$\sum_{k=1}^{n} \lambda(r_k + V) = \sum_{k=1}^{n} \lambda(V) = n\,\lambda(V)$$

So $\quad n\,\lambda(V) \le 6 \quad \forall n.$

But $\lambda(V) > 0$ gives $\times$.

$\square$