REVIEW OF RIEMANN INTEGRAL [A, 1.A]

DEF LET $B \subseteq \mathbb{R}$ be a BOUNDED set.

1. $M = \sup(B)$ is the supremum or Least Upper Bound of B if

   a) M is an upper bound for B:

   $$\forall x \in B \qquad x \leq M$$

   b) $\forall \varepsilon > 0$, $M - \varepsilon$ is NOT an U.B. for B:

   $$\exists\, x_* \in B : \quad x_* > M - \varepsilon$$

2. Similarly we define $m = \inf B$ as the infimum or Greatest Lower Bound.

DEF LET $f : \mathbb{R} \to \mathbb{R}$ be bounded, $A \subseteq \mathbb{R}$.

Then

$$\sup_A f = \sup \{ f(x) \,/\, x \in A \}$$

$$\inf_A f = \inf \{ f(x) \,/\, x \in A \}.$$

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DEF A PARTITION of $[a,b]$ is finite list

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

DEF LET $f : [a,b] \to \mathbb{R}$ be bounded and

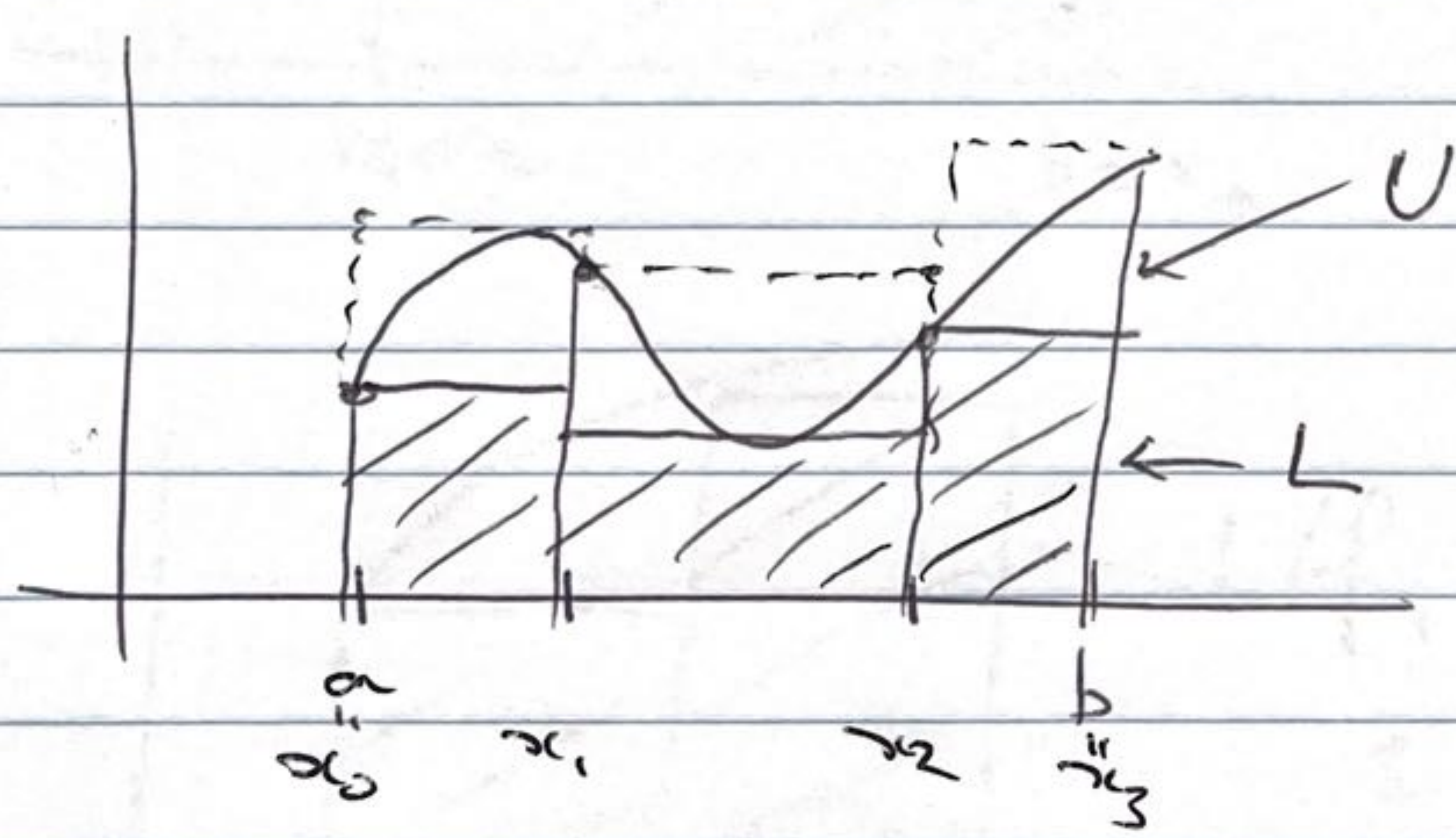
$P = \{x_0, \dots, x_n\}$ a partition of $[a,b]$.

Define

$$L(f, P, [a,b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

$$U(f, P, [a,b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f$$

LOWER + UPPER DARBOUX SUMS.



DEF A partition P' is a REFINEMENT of P

if $$P' = P \cup Q \qquad (P \subseteq P')$$

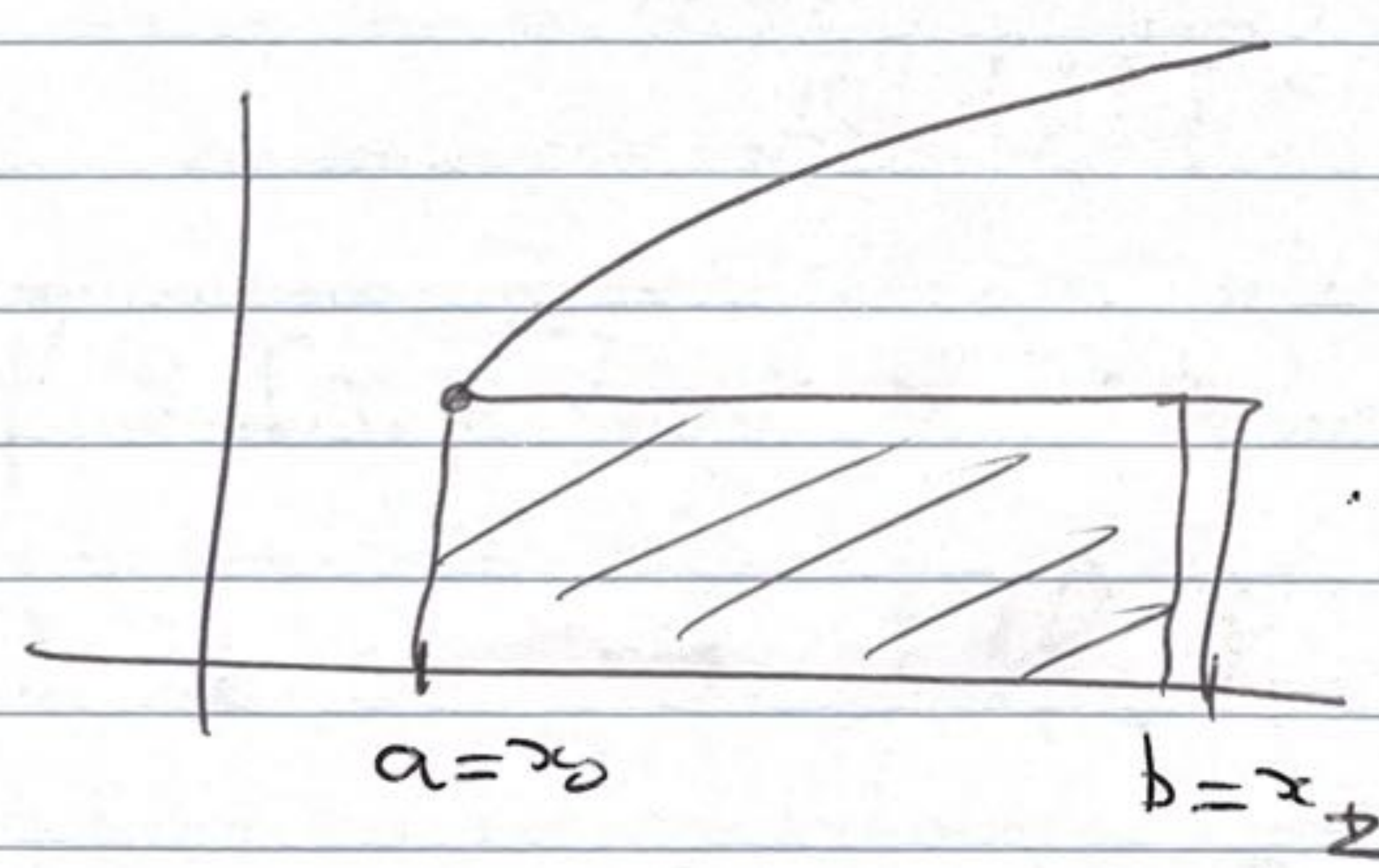
for some set of points Q.

PROP 1
When you refine a partition

The Lower Darboux Sum increases

The Upper Darboux Sum decreases

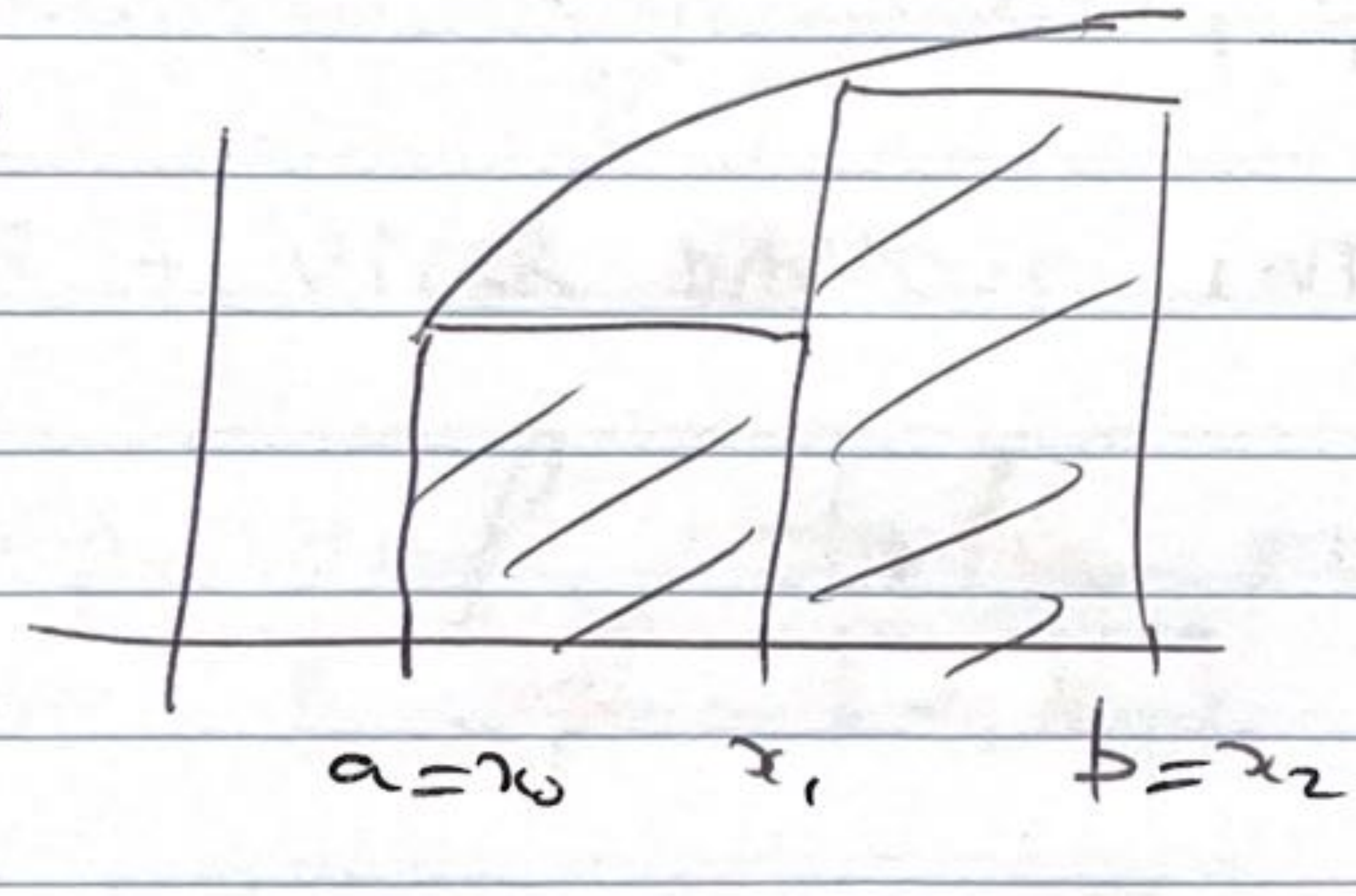
$P = \{x_0, x_2\}$



$L(f, P, [a, b])$

SMALLER

$P' = \{x_0, x_1, x_2\}$



$L(f, P', [a, b])$

BIGGER

PROP 2 Let $f : [a,b] \to \mathbb{R}$ be bounded and P, P' partitions of $[a,b]$

Then

$$L(f, P, [a,b]) \leq U(f, P', [a,b])$$

PF Let $P'' = P \cup P'$

Then by PROP 1

$$L(f, P, [a,b]) \overset{\text{PROP 1}}{\leq} L(f, P'', [a,b])$$

$$\overset{\text{INF} \leq \text{SUP}}{\leq} U(f, P'', [a,b])$$

$$\overset{\text{PROP 1}}{\leq} U(f, P, [a,b])$$

DEF Let $f : [a,b] \to \mathbb{R}$ be bounded.

$$L(f, [a,b]) = \sup_P L(f, P, [a,b])$$

$$U(f, [a,b]) = \inf_P U(f, P, [a,b])$$

are LOWER + UPPER DARBOUX INTEGRALS of f.

Here sup, inf taken over all partitions P of $[a,b]$.

NOTE By Prop 2

The Set $\{ L(f, P, [a,b]) \mid P \text{ is partition of } [a,b] \}$

is bounded above by

$$U(f, \{x_0, x_1\}, [a,b]) = \left(\sup_{[a,b]} f\right)(b-a)$$

So $L(f,[a,b])$ $\exists$ and is finite.

PROP 3 $\quad L(f,[a,b]) \leq U(f,[a,b])$

PF LET $\varepsilon > 0$. Since $L(f,[a,b])$ is a LUB $\exists P$:

$$\underbrace{L(f,[a,b]) - \varepsilon}_{\text{NOT U.B.}} < L(f, P, [a,b])$$

Similarly $\exists P'$:

$$U(f,[a,b]) + \varepsilon > U(f, P', [a,b])$$

So by Prop 2

$$L(f,[a,b]) - \varepsilon < L(f,P,[a,b]) \leq U(f, P', [a,b]) < U(f,[a,b]) + \varepsilon$$

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Since this is true $\forall \varepsilon > 0$

$$L(f, [a,b]) \leq U(f, [a,b]). \qquad \square$$

DEF Let $f : [a,b] \to \mathbb{R}$ be bounded. We say f is RIEMANN INTEGRABLE if

$$L(f, [a,b]) = U(f, [a,b])$$

in which case

$$\int_a^b f \, dx := L(f, [a,b]) = U(f, [a,b]).$$

THM Let $f : [a,b] \to \mathbb{R}$ be CTS on bounded interval. Then f is Riemann Integrable

STS $U(f, [a,b]) \leq L(f, [a,b])$

PF LET $\varepsilon > 0$. Since f is UNIFORMLY CTS

$$\exists \delta > 0 : |s - t| < \delta \overset{(*)}{\Rightarrow} |f(s) - f(t)| < \varepsilon.$$

Choose $n : \Delta x \equiv \frac{b-a}{n} < \delta$ and let

$P = \{x_0, \dots, x_n\}$ where $x_k = a + k\Delta x$

Then

$$\underset{GLB}{U}(f, [a,b]) - \underset{LUB}{L}(f, [a,b])$$

$$\leq U(f, P, [a,b]) - L(f, P, [a,b])$$

$$= \frac{b-a}{n} \sum_{j=1}^{n} \left(\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f \right)$$

$$\leq \frac{b-a}{n} \sum_{j=1}^{n} [f(\xi_j) - f(\eta_j)]$$

as f is CTS so sup + inf are attained

at some $\xi_j, \eta_j \in [x_{j-1}, x_j]$

$$\leq \frac{b-a}{n} \cdot \sum_{j=1}^{n} \varepsilon \qquad \text{by } (\ast) \text{ as } |\xi_j - \eta_j| < \delta$$

$$= (b-a)\, \varepsilon$$

Since true $\forall \varepsilon > 0$: $U(f, [a,b]) \leq L(f, [a,b])$ $\square$

DEF Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$.

(a) $\|P\| = \max_j |x_j - x_{j-1}|$ DIAMETER of P.

(b) $\{\xi_j\}_{j=1}^n$ with $\xi_j \in [x_{j-1}, x_j]$

is a collection of TAGS for P

(In practice since we can't find sup, inf easily its better to just pick points ξ_j at which to evaluate f)

(c) RIEMANN SUM

$$\sigma(f, P, \xi, [a,b]) = \sum_{j=1}^n f(\xi_j)(x_j - x_{j-1})$$

THM Let $P^{(n)}$ be a sequence of partitions of $[a,b]$ with $\|P^{(n)}\| \to 0$ as $n \to \infty$.

Let $\xi^{(n)}$ be tags for $P^{(n)}$

Let $f : [a,b] \to \mathbb{R}$ be R.I.

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sigma(f, P^{(n)}, \xi^{(n)})$$

PF OMIT.