COURSE OVERVIEW

I LEBESGUE MEASURE

LET $A \subseteq \mathbb{R}^n$

- Want to define the <u>measure</u>, $\lambda(A)$, of A

  $n=1$: Length

  $n=2$: Area

  $n=3$: Volume

- Notion of measure should have nice properties

  EG If $\{A_k\}_{k=1}^{\infty}$ are disjoint sets

  Then $$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k).$$

- PROBLEM: If we want to define $\lambda(A)$ for all subsets $A \subset \mathbb{R}^n$ ~~This~~ it is impossible for all nice properties to hold.

- SOLUTION: Only define $\lambda(A)$ for a special class of <u>measurable</u> sets.

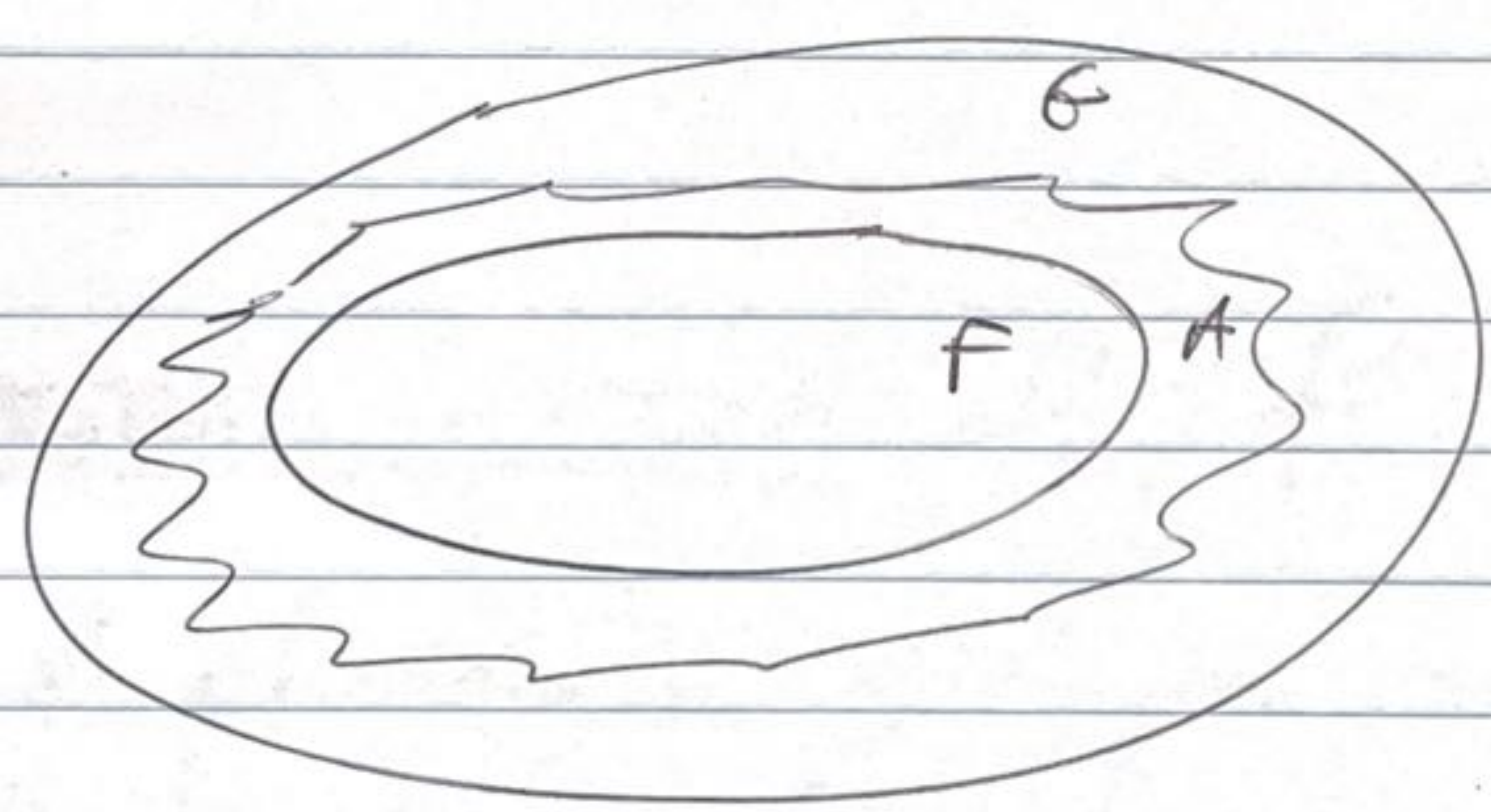
- There are several equivalent defns of a measurable set.

- We will construct Lebesgue measure in stages

- Polygons
- Open sets
- Compact sets
- General measurable sets.

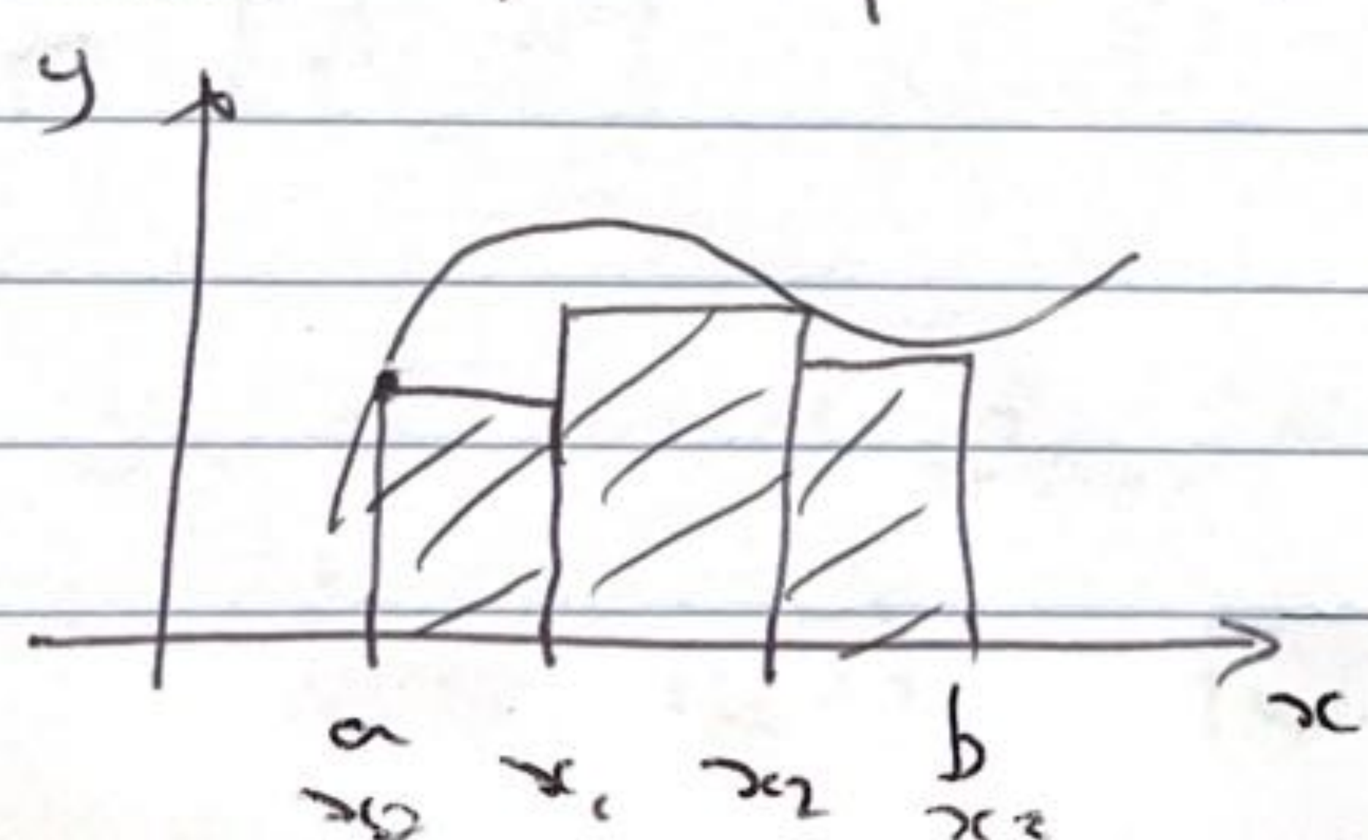
Then we will prove

$$A \text{ is measurable} \iff \begin{array}{l} \forall \varepsilon > 0 \\ \exists \text{ closed } F, \text{ Open } G \\ \text{with } F \subset A \subset G \\ \text{and } \lambda(G \sim F) < \varepsilon \end{array}$$



We will establish 12 nice properties of $\lambda(A)$.

(II) <u>RIEMANN INTEGRAL</u> of $f : [a,b] \to \mathbb{R}$



Approximate area under graph of f using a Riemann sum. (upper or lower)

If as make Δx smaller upper + lower sums both converge to same limit say f is Riemann integrable and $\int_a^b f\,dx$ denotes the common limit.

BUT Riemann integral has several problems.

So

III LEBESGUE INTEGRAL of $f : [a,b] \to \mathbb{R}$

Instead of choosing a grid on x-axis and using $y = f(x)$ to give heights of rectangles

Choose grid on y-axis and use f to give widths of rectangles

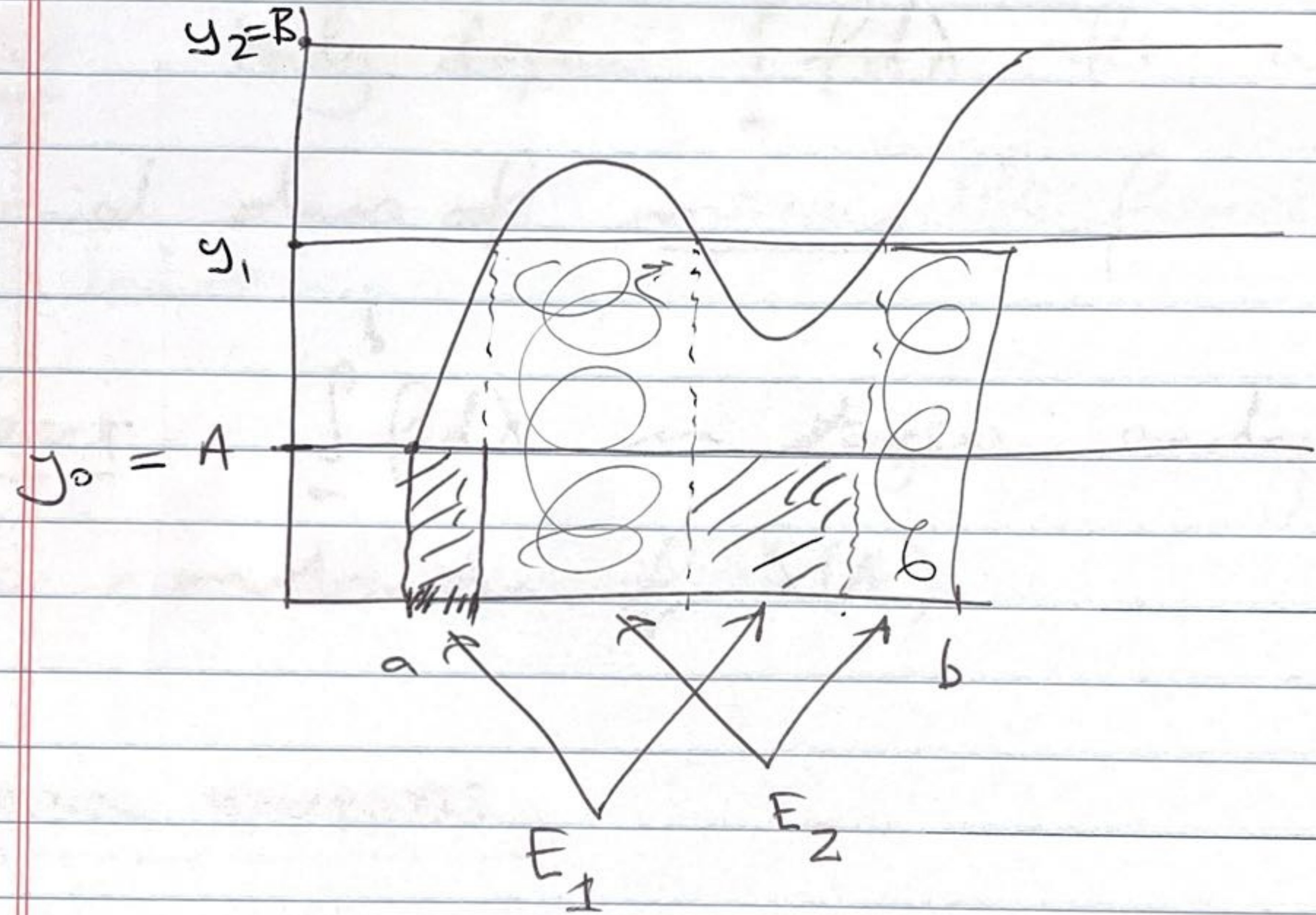
So If $f([a,b]) \subseteq [A\ B]$ pick

$$A = y_0 < y_1 < \dots < y_{n-1} < y_n = B$$

Let $E_k = f^{-1}([y_{k-1}, y_k])$ Inverse Image.

Define $\int_a^b f\,d\lambda = \lim_{n\to\infty} \sum_{k=1}^{n} y_{k-1}\,\lambda(E_k)$

HEIGHT (y_{k-1}) WIDTH ($\lambda(E_k)$)



$E_1 = f^{-1}([y_0, y_1])$

$E_2 = f^{-1}([y_1, y_2])$

$\int_a^b f\,d\lambda \simeq$ Area + Area

PROBLEM $\exists$ functions that are not Lebesgue Integrable (whatever that means)

SOLUTION: Only define $\int_a^b f\,d\lambda$ for a special class of measurable functions

CONSTRUCT $\int_a^b f\,d\lambda$ in stages analogously to construction of $\lambda(A)$

CONVERGENCE THEOREMS

Suppose $f_n \to f$ pointwise.

Under what conditions on f_n do we have
$$\int_a^b f_n\,d\lambda \longrightarrow \int_a^b f\,d\lambda \quad ?$$

- FATOU'S LEMMA
- MONOTONE CONVERGENCE THM
- LEBESGUE DOMINATED CONVERGENCE THM.

IV CONVERGENCE OF MEASURABLE FUNCTIONS

Suppose $f_n, f : [a,b] \to \mathbb{R}$ are measurable

$\exists$ Several notions of limit $f_n \to f$.

- POINTWISE
- UNIFORM
- POINTWISE ALMOST EVERYWHERE
- CONVERGENCE IN MEASURE.

What are relationships between these notions?

EX EGOROV'S THM Let f_n, f be measurable.

Suppose $f_n \to f$ POINTWISE on $[a,b]$

Then $\forall \varepsilon > 0 \quad \exists E \subset [a,b]$ with $\lambda([a,b] \sim E) < \varepsilon$

and $f_n \to f$ UNIFORMLY on E.

HOW BAD CAN A MEASURABLE FUNCTION BE?

LUZIN'S THM Let $f : [a,b] \to \mathbb{R}$ be measurable.

$\forall \varepsilon > 0$ Can change values of f on open set of measure $< \varepsilon$ to create a continuous function.

V L^p SPACES

LET $0 < p \le \infty$

DEF $L^p(\mathbb{R})$ is set of measurable $f: \mathbb{R} \to \mathbb{R}$

So that
$$\|f\|_p := \left(\int_{\mathbb{R}} |f|^p \, d\lambda \right)^{\frac{1}{p}} < \infty$$

MAIN RESULTS

1. $L^p(\mathbb{R})$ is a normed vector space
2. $L^p(\mathbb{R})$ is complete, so it is a Banach space
3. MINKOWSKI INEQUALITY (Δ)
$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$
4. HÖLDER'S INEQUALITY

If $\frac{1}{p} + \frac{1}{q} = 1$ ~~Then~~ and $f \in L^p, g \in L^q$
Then $fg \in L^1$ and
$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

VI FTC WORKS FOR LEBESGUE INTEGRAL