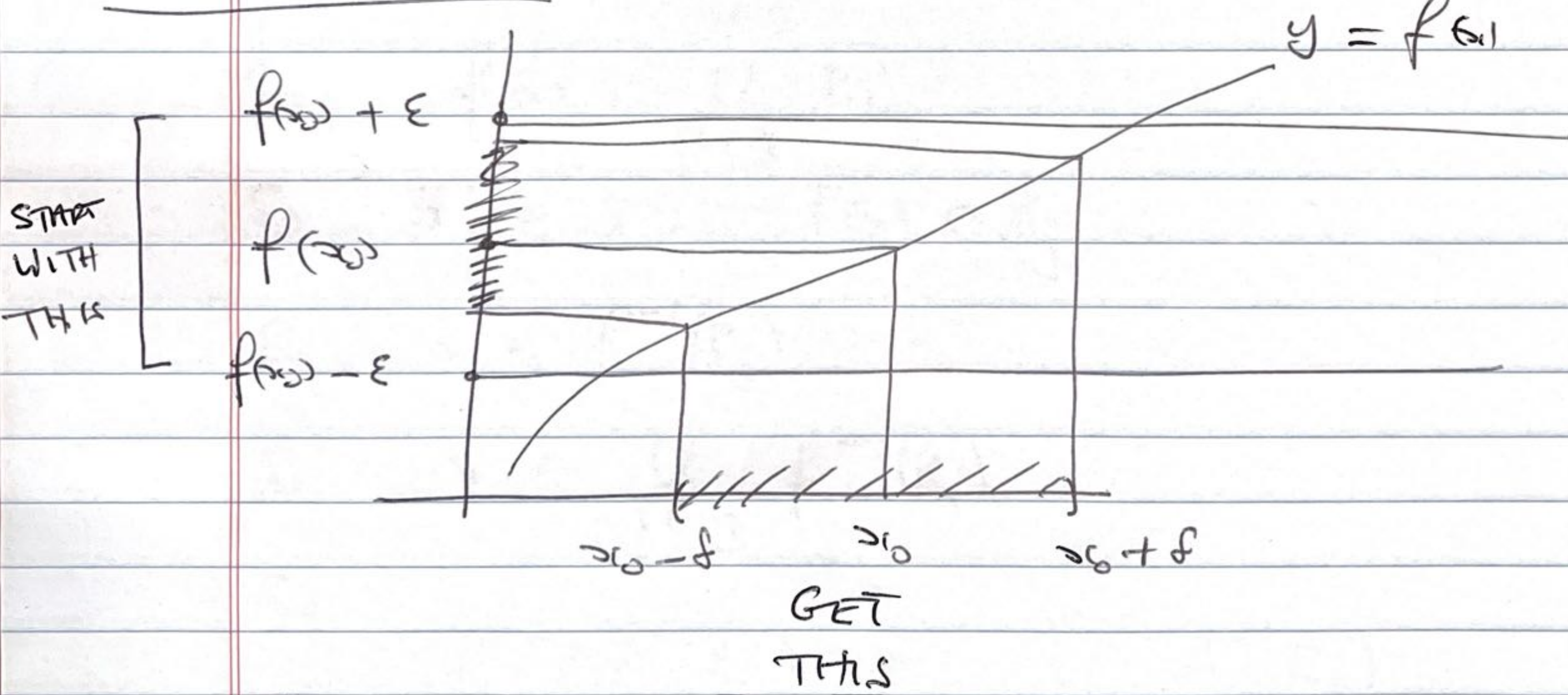
## CONTINUOUS FUNCTIONS [J, 1.E]

**DEF** Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ is CTS at $x_0 \in A$ if

$\forall \varepsilon > 0 \; \exists \delta > 0$ : whenever $\|x - x_0\| < \delta$ we have that

$$\|f(x) - f(x_0)\| < \varepsilon$$

PICTURE $(n = m = 1)$




**DEF** $f : \mathbb{R}^n \to \mathbb{R}^m$

ⓐ If $A \subseteq \mathbb{R}^n$ The IMAGE of $A$ is $f(A) = \{f(x) \,/\, x \in A\}$

ⓑ If $B \subseteq \mathbb{R}^m$ The ~~PRE~~ INVERSE IMAGE of $B$ is $f^{-1}(B) = \{x \in \mathbb{R}^n \,/\, f(x) \in B\}$

PROPERTIES FOR ANY INDEX SET $I$

a) $$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

b) $$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$$

PROOF OF a)

$\boxed{\subseteq}$ LET $x \in f^{-1}\left(\bigcup_{i \in I} B_i\right)$

So $f(x) \in \bigcup_{i \in I} B_i$

So $\exists j \in I : f(x) \in B_j$

So $x \in f^{-1}(B_j)$

So $x \in \bigcup_{i \in I} f^{-1}(B_i)$

So

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) \subseteq \bigcup_{i \in I} f^{-1}(B_i)$$

$\boxed{\supseteq}$ YOU DO IT!

**THM** LET $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$.

$f$ CTS on $A \iff \forall$ open $G \subseteq \mathbb{R}^m$ $\exists$ open $H \subseteq \mathbb{R}^n$:

$$f^{-1}(G) = A \cap H$$

NOTE If $f : \mathbb{R}^n \to \mathbb{R}^m$ this condition just says the Inverse Image of every open set in $\mathbb{R}^m$ is an open set in $\mathbb{R}^n$.

PF JUST DO CASE $A = \mathbb{R}^n$.

($\Leftarrow$) Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$.

Now $B(f(x_0), \varepsilon) \subseteq \mathbb{R}^m$ is open in $\mathbb{R}^m$

So by assump$^n$ $f^{-1}(B(f(x_0), \varepsilon))$ is open in $\mathbb{R}^n$.

Now $x_0 \in f^{-1}(B(f(x_0), \varepsilon))$ as $f(x_0) \in B(f(x_0), \varepsilon)$

So $\exists \delta > 0$ : $B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$ by def$^n$ open

Let $\underline{\|x - x_0\| < \delta.}$

Then $x \in B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon))$

So $f(x) \in B(f(x_0), \varepsilon)$

i.e. $\underline{\|f(x) - f(x_0)\| < \varepsilon}$

The underlined statement says $f$ is CTS at $x_0$.

$\boxed{\Rightarrow}$ Suppose $f$ is CTS on $\mathbb{R}^n$.
LET $G \subseteq \mathbb{R}^m$ be open.
Must show $f^{-1}(G)$ is open

Let $x \in f^{-1}(G)$

Since $f(x) \in G$ which is open $\exists \varepsilon(x) > 0$:

$$B(f(x), \varepsilon(x)) \subset G$$

Using this $\varepsilon(x)$ and def$^n$ of CTY of $f$

$$\exists \delta(x) > 0 : \|x - y\| < \delta(x) \overset{*}{\Rightarrow} \|f(x) - f(y)\| < \varepsilon(x)$$

LET

$$H := \bigcup_{x \in f^{-1}(G)} B(x, \delta(x)) \subseteq \mathbb{R}^n.$$

H is OPEN !!

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CLAIM $H = f^{-1}(G)$

PF $\boxed{\supseteq}$ ✓

$\boxed{\subseteq}$ Let $y \in H$

Then $\exists\, x \in f^{-1}(G) : \|y - x\| < \delta(x)$

So by (*) $\|f(y) - f(x)\| < \varepsilon(x)$

So $f(y) \in B(f(x), \varepsilon(x)) \subset G$

So $y \in f^{-1}(G)$ $\square$

THM [J, p17]

Let $A \subseteq \mathbb{R}^n$, $f : A \xrightarrow{CTS} \mathbb{R}^m$, $K$ compact in $A$

Then $f(K)$ is compact

CTS IMAGE of a compact set is compact

**DEF** $f : A \to \mathbb{R}^m$ is UNIFORMLY CTS on $A$
$A \subseteq \mathbb{R}^n$

if $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x, y \in A$

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon$$

**NOTE** CTY : $\delta = \delta(x)$
UNIFORM CTY : $\delta$ INDEPT of $x$.

**THM** Let $K \subseteq \mathbb{R}^n$ be compact, $f : K \xrightarrow{\text{CTS}} \mathbb{R}^m$

Then $f$ is UNIFORMLY CTS.

**PF** LET $\varepsilon > 0$. $\forall x \in K \ \exists \delta(x) > 0$ :

$$\|x - y\| < \delta(x) \implies \|f(x) - f(y)\| < \varepsilon/2$$

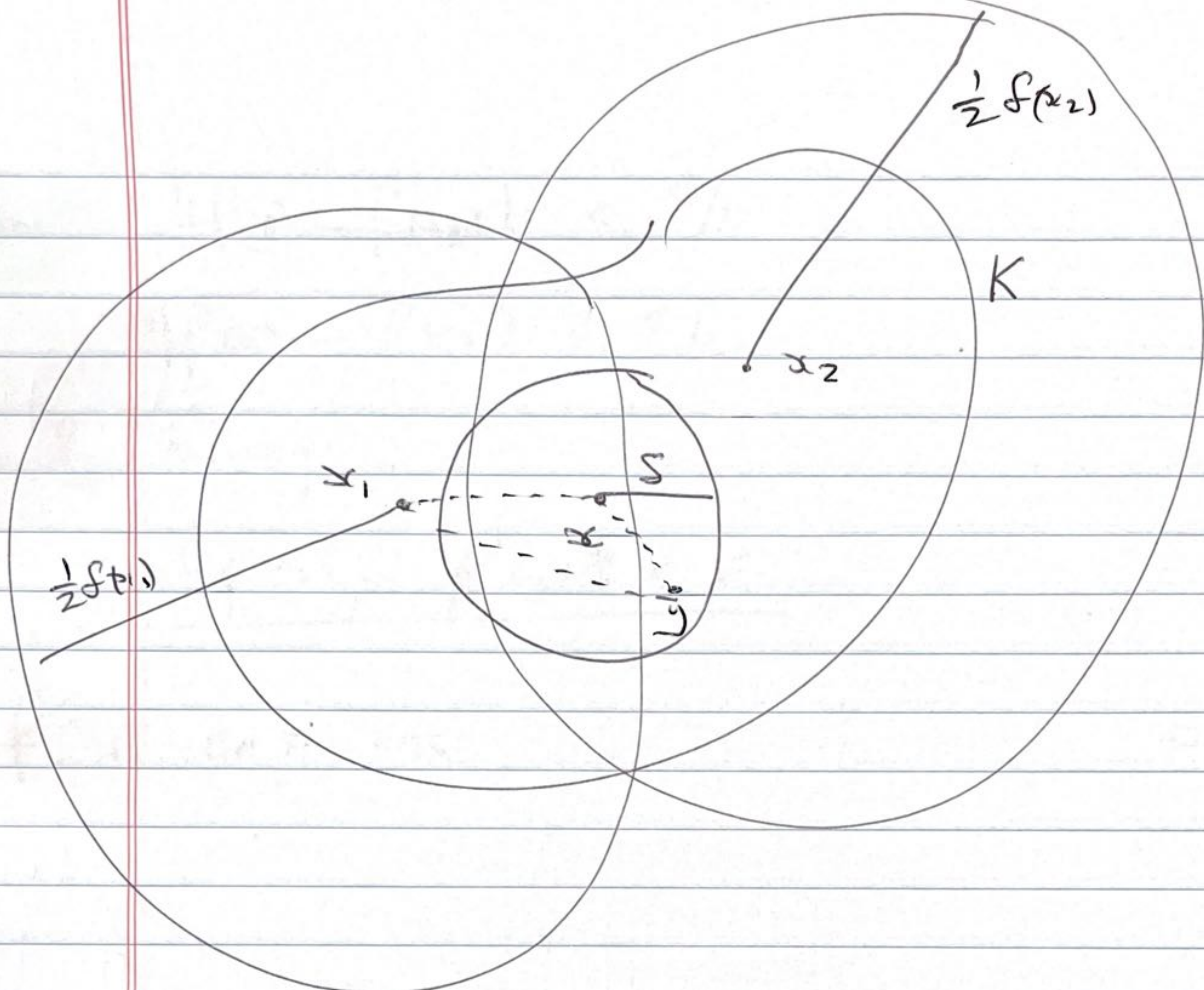
Clearly

$$K \subseteq \bigcup_{x \in K} B\left(x, \tfrac{1}{2}\delta(x)\right)$$

Since $K$ Cpt $\exists\, x_1, \dots, x_N \in K$ :

$$K \subseteq \bigcup_{k=1}^{N} B\left(x_k, \tfrac{1}{2}\delta(x_k)\right)$$

LET $\delta = \min\left\{\tfrac{1}{2}\delta(x_1), \dots, \tfrac{1}{2}\delta(x_N)\right\} > 0$ (+)




LET $x, y \in K$ with $\|x - y\| < \delta$

$\exists k$: $\quad x \in B(x_k, \frac{1}{2}\delta(x_k))$

So

$$\|y - x_k\| \leq \|y - x\| + \|x - x_k\| \qquad \Delta \text{ INEQ}$$

$$< \delta + \frac{1}{2}\delta(x_k)$$

$$< \delta(x_k) \qquad \text{by } (\dagger)$$

So $\quad y \in B(x_k, \delta(x_k))$

and $\quad x \in B(x_k, \delta(x_k))$

So $\|f(y) - f(x_k)\| < \varepsilon/2$

$\|f(x) - f(x_k)\| < \varepsilon/2$

So by $\Delta$ Ineq

$\|f(x) - f(y)\| < \varepsilon$

$\therefore f$ is UNIF CTS $\square$