FUBINI'S THEOREM FOR $\mathbb{R}^n$

[J, 8]

NO PROOFS

SET UP

$n = l + m$, $\quad \mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^m$
$\qquad\qquad\quad z = (x, y)$

If $f : \mathbb{R}^n \to \mathbb{R}$ define

$f_y : \mathbb{R}^l \to \mathbb{R}$ by $f_y(x) = f(x,y)$

$f_x : \mathbb{R}^m \to \mathbb{R}$ by $f_x(y) = f(x,y)$

TONELLI'S THM $[f \geq 0]$

Suppose $f : \mathbb{R}^n \to [0, \infty]$ is L-measurable

Then for a.e. $y \in \mathbb{R}^m$

f_y ~~is~~ $\mathbb{R}^l \to [0, \infty]$ is L-measurable

and so

$$F(y) := \int_{\mathbb{R}^l} f_y(x)\, dx$$

Then

$$\int_{\mathbb{R}^n} f(z)\, dz = \int_{\mathbb{R}^m} F(y)\, dy = \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^l} f(x,y)\, dx \right] dy$$

FUBINI'S THM

Suppose $f \in L^1(\mathbb{R}^n)$

Then a.e. $y \in \mathbb{R}^m$, $f_y \in L^1(\mathbb{R}^l)$ and

$$F(y) := \int_{\mathbb{R}^l} f_y(x)\,dx \quad \exists$$

Furthermore $F \in L^1(\mathbb{R}^m)$ and

$$\int_{\mathbb{R}^n} f(z)\,dz = \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^l} f(x,y)\,dx\right] dy$$

NOTE In practice to show $f \in L^1(\mathbb{R}^n)$

use Tonelli to show

$$\int_{\mathbb{R}^n} |f(z)|\,dz = \int_{\mathbb{R}^m} \int_{\mathbb{R}^l} |f(x,y)|\,dx\,dy < \infty.$$

Then you can use Fubini to calculate or estimate the integral

$$\int_{\mathbb{R}^n} f(z)\,dz$$

THM

Let $A \subseteq \mathbb{R}^l$, $B \subseteq \mathbb{R}^m$ be measurable

Then

$$A \times B \subseteq \mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^m \text{ is measurable}$$

and

$$\lambda(A \times B) = \lambda(A)\,\lambda(B)$$

PF IDEA

USE

$$A \text{ measurable} \iff \exists \underset{\text{CLOSED}}{F} \subset A \subset \underset{\text{OPEN}}{G} : \lambda(G \setminus F) < \varepsilon$$

to show $A \times B$ measurable.

Then apply Tonelli's Thm to

$$f(x,y) = \chi_{A\times B}(x,y) = \chi_A(x)\,\chi_B(y) \qquad \square$$

ANOTHER

USEFUL RESULT ON IMPROPER INTEGRALS

Suppose $f : [a,\infty) \to \mathbb{R}$ is Riemann integrable on $[a, R]$ for all $R > a$. Then

$$f \in L^1([a,\infty)) \iff \int_a^\infty |f(x)|\,dx := \lim_{R\to\infty} \int_a^R |f(x)|\,dx \;\exists$$

and

$$\int_{[a,\infty)} f\,d\lambda = \int_a^\infty |f(x)|\,dx$$

EXS

(1) LET $f(x,y) = \sin x \, e^{-xy}$

Then $f \in L^1((0,a) \times (0,\infty))$

PF

Since $\left|\frac{\sin x}{x}\right| \leq 1 \ \forall x$ we have

$|f(x,y)| \leq \underbrace{x e^{-xy}}_{\text{Measurable}}$. on $(0,a) \times (0,\infty)$.

So by Tonelli and Useful Result

$$\int_{(0,a)\times(0,\infty)} |f| \, dx dy \leq \int_0^a \left[\int_0^\infty x e^{-xy} \, dy\right] dx$$

$$= \int_0^a \left[-e^{-xy}\right]_{y=0}^{y=\infty} dx$$

$$= \int_0^a e^0 \, dx = a. < \infty$$

So $f \in L^1$.

(2) Now Apply Fubini to $f(x,y) = \sin x \, e^{-xy}$ to calculate

$$\int_0^a \frac{\sin x}{x} \, dx = \int_{x=0}^{a} \int_{y=0}^{\infty} \sin x \, e^{-xy} \, dy \, dx$$

$$\overset{\text{FUBINI}}{=} \int_{y=0}^{\infty} \int_{x=0}^{a} \sin x \, e^{-xy} \, dx \, dy$$

$$= \frac{\pi}{2} - \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin a \int_0^\infty \frac{y e^{-ay}}{1+y^2} dy$$

Hence

$$\int_0^\infty \frac{\sin x}{x} dx \quad \lim_{a \to \infty} \int_0^a \frac{\sin x}{x} dx = \pi/2 \quad \text{holds}$$

(3) LET $f(x,y) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

Can show

$$\int_{y=0}^{1} \int_{x=0}^{1} f(x,y) \, dx \, dy = -\pi/4 \qquad (*)$$

and so $\displaystyle\int_{x=0}^{1} \int_{y=0}^{1} f(x,y) \, dy \, dx = +\pi/4$

HINT FOR (*)

$$\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

probably helps do $\displaystyle\int_{x=0}^{1} f(x,y) \, dx$

This show $f \notin L^1([0,1]\times[0,1])$

Can we show that directly??

Well

$$|f(r,\theta)| = \frac{|\cos 2\theta|}{r^2} \geq \chi_{\{\theta<\theta_0\}}(\theta)\,\frac{\cos(2\theta_0)}{r^2}$$

for any $\theta_0 < \pi/4$ $\quad \cos(2\theta_0) > \cos \pi/2 = 0$

So

$$\int_{[0,1]\times[0,1]} |f|\,dx\,dy \geq \int_{\theta<\theta_0} \int_{r=0}^{1} \frac{\cos(2\theta_0)}{r^2}\, r\,dr\,d\theta$$

$$= \theta_0 \cos(2\theta_0) \int_0^1 \frac{dr}{r} = \infty$$

So

$f \notin L^1([0,1]\times[0,1])$