ABSTRACT MEASURE SPACES [J, 6, F]

<u>DEF^n 1</u> A <u>MEASURE SPACE</u> $(X, \mathcal{M}, \mu)$ consists of

(a) A set X

(b) A σ-algebra $\mathcal{M} \subset 2^X$

(c) A function $\mu : \mathcal{M} \to [0, \infty]$ so that

(i) $\mu(\emptyset) = 0$

(ii) IF $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$ are DISJOINT

THEN
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

We call μ a <u>MEASURE</u> on X

<u>DEF 2</u> (a) If $s = \sum_{k=1}^{m} \alpha_k \chi_{A_k}$ is a simple, <u>$\mathcal{M}$-measurable</u> function on X

Then
$$\int_X s \, d\mu := \sum_{k=1}^{m} \alpha_k \, \mu(A_k)$$

(b) If $f : X \to [0, \infty]$ is $\mathcal{M}$-measurable we define

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu \,/\, \begin{array}{l} 0 \le s \le f \\ s \text{ SIMPLE} \\ \mu\text{-MEASURABLE} \end{array} \right\}$$

(c) If $f : X \to [-\infty, \infty]$ is $\mathcal{M}$-measurable we define

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu$$

if $\int_X f_{\pm} \, d\mu < \infty$.

In This case we call f <u>μ-Integrable</u>

ALL THEOREMS we stated ~~for~~ before still hold.

<u>NOTE</u> In This context by a <u>NULL</u> set we mean a set N with $\mu(N) = 0$

<u>EX 1</u> $(\mathbb{R}^n, \mathcal{L}, \lambda)$, $(\mathbb{R}^n, \mathcal{B}, \lambda)$

<u>EX2</u> <u>COUNTING MEASURE</u>

X = Any set

$\mathcal{M} = 2^X$

Define

$$\mu(A) = \begin{cases} \# \text{PTS IN } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

~~DEF~~ ~~Let~~

NOTE Since $\mathcal{M} = 2^X$ every function $f : X \to [-\infty, \infty]$ is $\mathcal{M}$-measurable.

DEF Let $f : X \to [0, \infty]$. Define

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \;\middle|\; F \subset X \text{ is FINITE SET} \right\}$$

PROP ① If $f : X \to [0, \infty]$ Then

$$\int f \, d\mu = \sum_{x \in X} f(x)$$

② If $f : X \to [-\infty, \infty]$ Then

$$f \in L^1 \iff \sum_{x \in X} |f(x)| < \infty.$$

NOTE IF $X = \mathbb{N}$ Then

$$\int_{\mathbb{N}} f \, d\mu = \lim_{n \to \infty} \sum_{k=1}^{n} f(k) = \sum_{k=1}^{\infty} f(k)$$

LDCT FOR COUNTING MEASURE

For each $k = 1, 2, 3, \ldots$ suppose

$\{a_n^{(k)}\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$

so that

① $a_n := \lim_{k \to \infty} a_n^{(k)}$ $\exists$

② $\exists$ ~~convergent~~ sequence $\{b_n\}_{n=1}^{\infty}$

with $b_n \geq 0$ $\forall n$ and $\sum_{n=1}^{\infty} b_n < \infty$

so that
$$|a_n^{(k)}| \leq b_n \quad \forall k$$

Then

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} a_n^{(k)} = \sum_{n=1}^{\infty} a_n$$

EX3 WEIGHTED INTEGRALS

Given $(\mathbb{R}^n, \mathcal{L}, \lambda)$ and $h : \mathbb{R}^n \to [0, \infty]$ $\mathcal{L}$-measurable

Define
$$\nu(A) = \int_A h \, d\lambda \qquad \text{for } A \in \mathcal{M}$$

Then $(\mathbb{R}^n, \mathcal{L}, \nu)$ is a measure space

NOTE $\lambda(A) = 0 \Rightarrow \nu(A) = 0$ holds

LEMMA $\int s\,d\nu = \int s h\,d\lambda$ s simple

PF $s = \sum \alpha_k \chi_{A_k}$

$$\int s\,d\nu = \sum_{k=1}^{m} \alpha_k \, \nu(A_k)$$

$$= \sum_{k=1}^{m} \alpha_k \int_{A_k} h\,d\lambda$$

$$= \int_{\mathbb{R}^n} \left(\sum_{k=1}^{m} \alpha_k \chi_{A_k}\right) h\,d\lambda$$

$$= \int_{\mathbb{R}^n} s h\,d\lambda$$

PROP If $f: X \to [0,\infty]$ is ~~$\mathcal{L}$~~- measurable
Then
$$\int f\,d\nu = \int f h\,d\mu$$

SYMBOLICALLY: $d\nu = h\,d\mu$

OR $\frac{d\nu}{d\mu} = h$

PF $\exists\ 0 \le s_1 \le s_2 \le \dots \le f$, $\quad s_k \to f$

So $0 \le s_1 h \le s_2 h \le \dots \le fh \quad s_k h \to hf$

So

$$\int f \, d\nu \overset{MCT}{=} \lim_{k \to \infty} \int s_k \, d\nu$$

$$= \lim_{k \to \infty} \int s_k h \, d\lambda$$

$$\overset{MCT}{=} \int \lim_{k \to \infty} (s_k h) \, d\lambda = \int f \, d\lambda$$

$\square$