EXS + APPLICATIONS OF CONVERGENCE THMS

EX 1 LET $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$

- f is measurable
- $f_+ = \chi_{[0,\infty)}$ $\quad \int f_+ \, d\lambda = \lambda([0,\infty)) = \infty$

  $f_- = \chi_{(-\infty,0)}$ $\quad \int f_- \, d\lambda = \lambda((-\infty,0)) = \infty.$
- So $f \notin L^1$.

EX 2 $f(x) = \begin{cases} x^2 & x \in [0,1] \setminus \mathbb{Q} \\ 1 & x \in [0,1] \cap \mathbb{Q}. \end{cases}$

~~Find~~ Show $f \in L^1$ and find $\int_{[0,1]} f \, d\lambda$

LET $g(x) = x^2$

Now $f = g$ a.e. as $\lambda([0,1] \cap \mathbb{Q}) = 0$

~~By Def 8.8~~

So $\int_{[0,1]} f \, d\lambda = \int_{[0,1]} g \, d\lambda \overset{*}{=} \int_0^1 x^2 \, dx = 1/3.$

$*$ Here we use fact g CTS $\Rightarrow$ g R.I. and use FTC for RI

$\boxed{\text{EX 3}}$ Let $f_k = \chi_{[k,k+1]}$

We have $f_k(x) \to 0$ as $k \to \infty$. $\forall x \in \mathbb{R}$

We know $\int f_k \, d\lambda = \int \chi_{[k,k+1]} d\lambda$

$= \lambda([k, k+1])$

$= 1 \quad \forall k$

$\neq \int 0 \, d\lambda = 0$

So LDCT must fail.

<u>REASON</u> If $\exists g$: $|f_k(x)| \leq g(x)$ for all sufficiently large k

Then $\exists R$: $g(x) \geq 1$ for all $x > R$

So $\int g \, d\lambda \geq \int_{[R,\infty)} 1 \, d\lambda = \infty$.

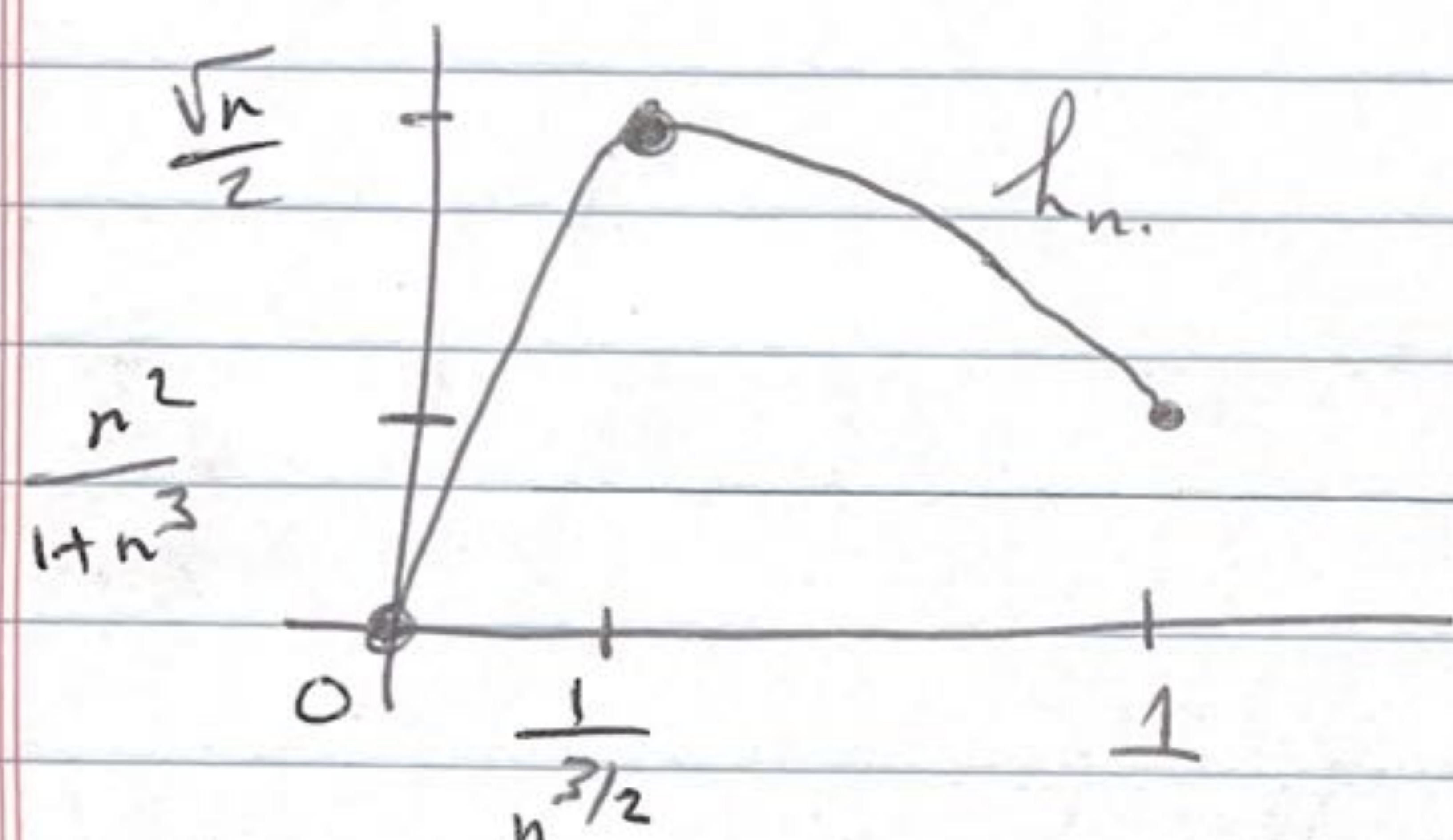
So $g \notin L^1$.

$\boxed{\text{EX 4}}$ [HISTORY NOTES PROBLEM 18]

Let

$$h_n(x) = \frac{n^2 x}{1+n^3x^2} \quad \text{on } [0,1],$$

CALCULUS SHOWS: CLAIM $h_n \to 0$ PW on $[0,1]$



So $\sup\limits_{[0,1]} h_n = \frac{\sqrt{n}}{2} \to \infty$ as $n \to \infty$.

So BCT does NOT apply

CLAIM (EX FOR YOU)

Let $g(x) = \frac{2^{2/3}}{3} x^{-1/3}$

Then

1. $h_n(x) \leq g(x) \quad \forall x \in (0,1]$

2. By Thm 2 from Riemann + Lebesgue Lecture

Can show

$$\int_{[0,1]} g\, d\lambda = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} g(x)\, dx = 2^{1/3}$$

So $g \in L^1([0,1])$ so a dominating f^n for h_n.

So

$$\lim_{n\to\infty} \int_{[0,1]} h_n \, d\lambda \overset{\text{LDCT}}{=} \int_{[0,1]} 0 \, d\lambda = 0.$$

<u>NOTE</u> We can explicitly calculate integral

$$\int_0^1 \frac{n^2 x}{1+n^3x^2}\, dx \qquad u = 1+n^3x^2$$
$$du = 2n^3 x\, dx$$

$$= \frac{1}{2n}\int_1^{1+n^3} \frac{du}{u}$$

$$= \frac{1}{2n} \log(1+n^3) \leq \frac{1}{2n}\log(2n^3)$$

$$= \frac{\log 2}{2n} + \frac{3\log n}{2n} \to 0 \text{ as } n \to \infty.$$

$\boxed{\text{EX 5}}$

Let $f_n(x) = (n+1)x^n$

SHOW
$$\int_{[0,1]} (\liminf_{n\to\infty} f_n)\, d\lambda < \liminf_{n\to\infty} \int_{[0,1]} f_n\, d\lambda$$

UPSHOT Do not always get $=$ in FATOU.

$\boxed{\text{RHS}}$
$$\int_{[0,1]} f_n\, d\lambda = \int_0^1 (n+1)x^n\, dx = \left[x^{n+1}\right]_0^1 = 1$$

So RHS $= 1$

$\boxed{\text{LHS}}$ LET $0 \le x < 1$

$$\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} (n+1)x^n = 0$$

as for $x < 1$ $n^2 x^n < 1$ for large enough n.

since if $a = \frac{1}{x} > 1$

Then
$$n^2 < a^n$$

EXPS GROW FASTER THAN POLYS as $y \to \infty$

$n \to y$
$$y^2 < a^y$$

So $(n+1)x^n < \dfrac{n+1}{n^2} \to 0$ as $n \to \infty$.

So $\boxed{\liminf f_n = 0 \text{ a.e.}}$.

LDCT GENERALIZATION

Suppose f_k are measurable, $f_k \to f$ PW a.e.

Suppose $g_k \in L^1$ with

$$|f_k| \leq g_k \quad \text{a.e.}$$

$$g_k \to g \text{ PW a.e.}, \quad g \in L^1$$

And $$\int g \, d\lambda = \lim_{k \to \infty} \int g_k \, d\lambda$$

Then $$\int f \, d\lambda = \lim_{k \to \infty} \int f_k \, d\lambda$$

EX 6

LET $F \in L^1(\mathbb{R})$

SHOW

$$\lim_{h \to 0} \int_{\mathbb{R}} |F(x+h) - F(x)| \, dx = 0$$

LET $h = \frac{1}{n}$, $\quad f_n = |F(x + \frac{1}{n}) - F(x)| \to 0$ PW

Now $$|f_n(x)| \leq |F(x+\tfrac{1}{n})| + |F(x)| =: g_n(x)$$

$F \in L^1$ implies $g_n \in L^1$ ~~dog~~ translation invariance ~~of~~ (See HWK)

Also $\quad g_n(x) \longrightarrow 2|F(x)| \quad$ as $\quad n \to \infty$

$$2|F(x)| = g(x)$$

And

$$\int g_n(x)\,dx = \int |F(x+\tfrac{1}{n})|\,dx + \int |F(x)|\,dx$$

$$= 2\int |F(x)|\,dx \quad \text{by translation invariance of } \int f\,d\lambda$$

$$= \int g(x)\,dx$$

So Assumptions of Generalized LDCT hold.

$\therefore$

$$\lim_{n\to\infty} \int_{\mathbb{R}} |F(x+\tfrac{1}{n}) - F(x)|\,dx$$

$$= \int_{\mathbb{R}} \lim_{n\to\infty} |F(x+\tfrac{1}{n}) - F(x)|\,dx$$

$$= \int_{\mathbb{R}} 0\,dx = 0$$

$\square$

DIFFERENTIATION UNDER INTEGRAL THM

Let $f = f(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

Suppose

① $\forall t$, $f_t(x) := f(x,t)$ has
$f_t \in L^1(\mathbb{R}^n)$

② f is differentiable wrt x

③ $\exists h \in L^1(\mathbb{R}^n) : \left|\frac{\partial f}{\partial t}(x,t)\right| \leq h(x)$
$\forall t$

Then

$$\frac{d}{dt} \int_{\mathbb{R}^n} f(x,t)\,dx = \int_{\mathbb{R}^n} \frac{\partial f}{\partial t}(x,t)\,dx$$

PF Let

$$F(t) = \int_{\mathbb{R}^n} f(x,t)\,dx$$

$$G_n(t) = \frac{F\left(t + \frac{1}{n}\right) - F(t)}{1/n}$$

$$= \int_{\mathbb{R}^n} \underbrace{\frac{f\left(x, t+\frac{1}{n}\right) - f(x,t)}{1/n}}_{g_n(x,t)} \, dx$$

Now

$$\frac{\partial f}{\partial t} = \lim_{n\to\infty} g_n.$$

$$|g_n(x,t)| \leq \left|\frac{\partial f}{\partial t}(x,t')\right| \quad \text{for some } t'$$

$$\overset{(3)}{\leq} |h(x)| \in L^1$$

So by LDCT

$$\frac{\partial F}{\partial t} = \lim_{n\to\infty} f_n(t)$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}^n} g_n(x,t)\, dx$$

$$= \int_{\mathbb{R}^n} \lim_{n\to\infty} g_n(x,t)\, dx$$

$$= \int_{\mathbb{R}^n} \frac{\partial f}{\partial t}(x,t)\, dx$$