RELATIONSHIP BETWEEN RIEMANN + LEBESGUE INTEGRALS

THM 1 LET $f : [a,b] \to \mathbb{R}$ be bounded.

(A) f is Riemann Integrable

$$\iff \lambda\left(\{ x \in [a,b] \mid f \text{ NOT CTS AT } x \} \right) = 0.$$

(B) If f is R.I. Then f is Lebesgue Integrable and

$$\underbrace{\int_a^b f(x)\,dx}_{\text{RIEMANN INTEGRAL}} = \int_{[a,b]} f \, d\lambda.$$

NOTE This result together with FTC for R.I. allows us to calculate Lebesgue Integrals of Continuous functions (for example)

PROOF

(A) Let P_n be partition of $[a,b]$ into 2^n subintervals $I_1 \dots I_{2^n}$ of equal size.

We assume each I_j is a CLOSED interval.

Let

$$g_n = \sum_{j=1}^{2^n} \left(\inf_{I_j} f\right) \chi_{I_j}$$

$$h_n = \sum_{j=1}^{2^n} \left(\sup_{I_j} f\right) \chi_{I_j}$$

SIMPLE MEASURABLE FUNCTIONS.

<u>CLAIM I</u> (UB✓)

$$L(f, P_n, [a,b]) = \int_{[a,b]} g_n \, d\lambda$$

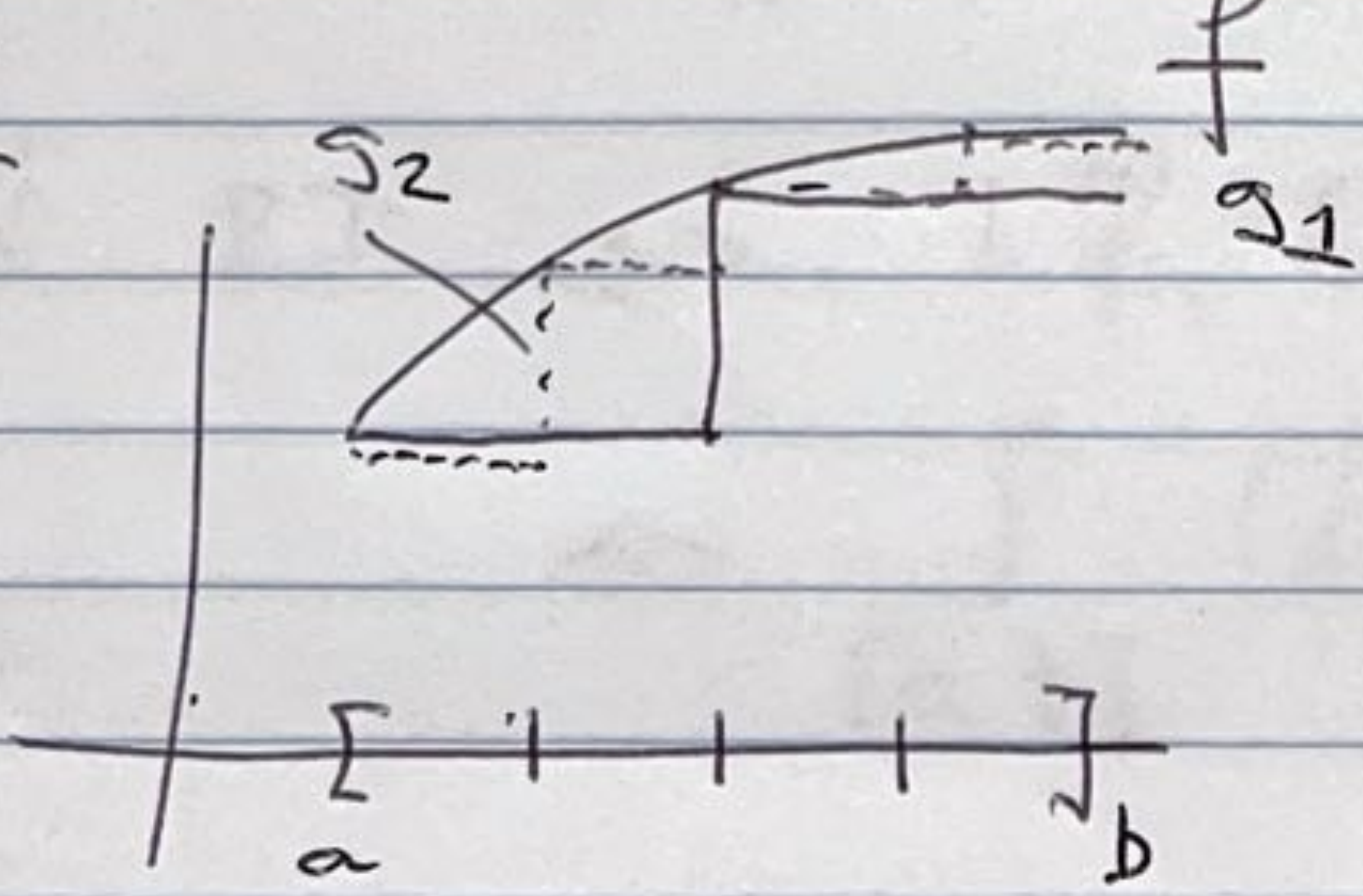
$$U(f, P_n, [a,b]) = \int_{[a,b]} h_n \, d\lambda$$

<u>CLAIM II</u>

$$g_1 \leq g_2 \leq \dots \leq f$$

$$h_1 \geq h_2 \geq \dots \geq f$$

<u>PICTURE PROOF</u>



g_1 has 2 Intervals
g_2 has 4 Intervals
obtained by subdividing those for g_1.

Define

$$f_L(x) = \lim_{n\to\infty} g_n(x) = \sup_n g_n(x)$$

$$f_U(x) = \lim_{n\to\infty} h_n(x) = \inf_n h_n(x).$$

By MCT

$$\int f_L \, d\lambda = \lim_{n\to\infty} \int g_n \, d\lambda = \lim_{n\to\infty} L(f, P_n, [a,b])$$

$$\overset{EX}{=} L(f, [a,b]) \quad \text{LOWER DARBOUX INTEGRAL}$$

By BCT since $|h_n| \le \sup_{[a,b]}(f) \quad \forall n$

$$\int f_U \, d\lambda = \lim_{n\to\infty} \int h_n \, d\lambda = \lim_{n\to\infty} U(f, P_n, [a,b])$$

$$\overset{EX}{=} U(f, [a,b]) \quad \text{UPPER DARBOUX INTEGRAL}$$

Recall: f RI $\iff L(f, [a,b]) = U(f, [a,b])$

$$\iff \int_{[a,b]} (f_U - f_L) \, d\lambda = 0$$

NOW by construction

$$f_L \le f \le f_U \Rightarrow \boxed{f_U - f_L \ge 0}$$

So
$$f \text{ RI} \Leftrightarrow \int_{[a,b]} (f_U - f_L)\, d\lambda = 0$$

$$\Leftrightarrow f_U = f_L \quad \text{a.e.}$$

$$\Leftrightarrow \lambda\left(\{x \in [a,b] \mid f_U(x) \neq f_L(x)\}\right) = 0$$

Let P_n be the 2^n points in nth partition and

$$P = \bigcup_{n=1}^{\infty} P_n. \qquad \lambda(P) = 0 \checkmark$$

NOW

$$\{x \in [a,b] \mid f_U(x) \neq f_L(x)\}$$

$$\subseteq \{x \in [a,b] \sim P \mid f_U(x) \neq f_L(x)\} \cup P$$

$$\overset{*}{\subseteq} \{x \in [a,b] \sim P \mid f \text{ NOT CTS AT } x\} \cup P$$

$$\subseteq \{x \in [a,b] \mid f \text{ NOT CTS at } x\} \cup P.$$

So $\lambda\left(\{x \in [a,b] \mid f_U(x) \neq f_L(x)\}\right) \leq \lambda\left(\{x \in [a,b] \mid f \text{ NOT CTS at } x\}\right)$

Hence f CTS a.e. $\Rightarrow f$ RI.

Similarly f RI $\Rightarrow f$ CTS a.e.

$\boxed{\text{To prove } (+)}$

Suppose $x \in [a,b] \sim P$ and $f_U(x) \neq f_L(x)$.

We must show f not cts at x.

So must show

$$\exists\, \varepsilon > 0 \;:\; \forall\, \delta > 0 \;\; \exists\, y \text{ with } |x-y| < \delta$$

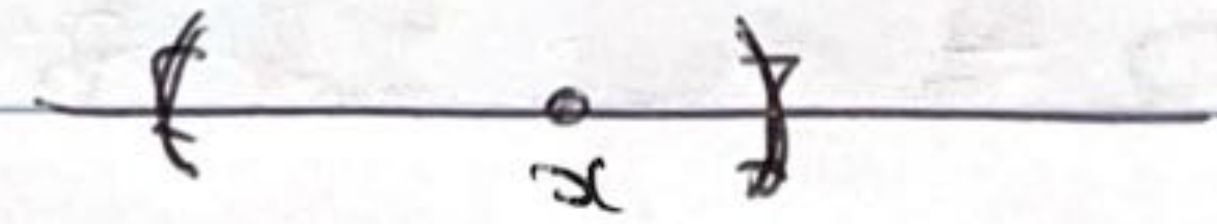
$$\text{so that } \quad |f(x) - f(y)| \geq \varepsilon .$$

Now let $\varepsilon = f_U(x) - f_L(x) > 0$.

Given any $\delta > 0$ choose n so that at level n:

$$\lambda(I_j) < \delta \qquad \forall j$$

We have $\exists j : x \in I_j^{\circ}$ as $x \notin P$.



Now for this level n

$$h_n(x) - g_n(x) > f_U(x) - f_L(x) > \varepsilon$$

So

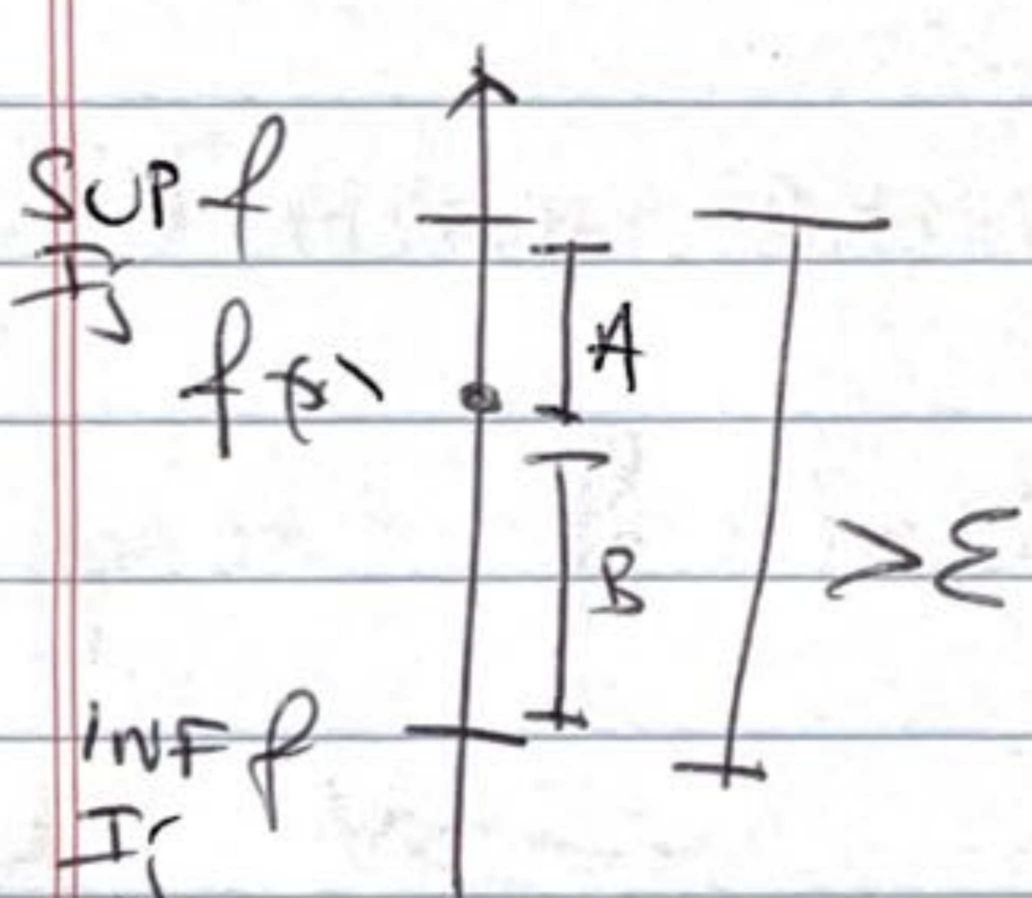
$$\sup_{I_j} f - \inf_{I_j} f > \varepsilon$$

as $h_n(x) = \sup_{I_j} f$ if $x \in I_j$

So $\exists\, y \in I_j$ i.e. $|x-y| < \delta$,

So that

$$|f(y) - f(x)| > \varepsilon/2$$



At least one of A, B must be bigger than $\varepsilon/2$.

<u>EXERCISE</u> Fill in details to show

$$f \text{ RI} \Rightarrow f \text{ CTS a.e}$$

(B) FINALLY f RI $\Rightarrow f$ CTS a.e.

$$\Rightarrow f \in L^1$$

Also f RI $\Rightarrow$ $L(f, [a,b]) = U(f, [a,b]) = \int_a^b f\,dx$

$$\int_{[a,b]} f_L\,d\lambda = \int_{[a,b]} f_U\,d\lambda = \int_{[a,b]} f\,d\lambda$$

so $f_U = f_L = f$ a.e.

$\square$

THM 2 [IMPROPER RIEMANN INTEGRALS]

Suppose $f : [a,b] \to [0, +\infty]$ is measurable

and $\forall \varepsilon > 0$ f is bounded on $[a+\varepsilon, b]$

and f is RI on $[a+\varepsilon, b]$

If $\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)\,dx =: \int_a^b f(x)\,dx$ $\exists$

Then $f \in L^1[a,b]$ and

$$\int_{[a,b]} f\,d\lambda = \int_a^b f(x)\,dx$$

PF HWK