LEBESGUE INTEGRAL OF A GENERAL MEASURABLE FUNCTION

[J, 6.B]

DEF 1 Let $f : \mathbb{R}^n \to [-\infty, +\infty]$ be measurable

Since $f_\pm$ are measurable with $f_\pm \geq 0$

$\int f_\pm \, d\lambda$ are defined.

If $\int f_\pm \, d\lambda < \infty$ Then f is Lebesgue Integrable and define Lebesgue Integral of f by

$$\int f \, d\lambda := \int f_+ \, d\lambda - \int f_- \, d\lambda$$

NOTATION

$$L^1(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R} \text{ M'ble so that } f \text{ is Lebesgue Integrable} \}$$

PROP 2 (HWK)

$$f \in L^1 \iff |f| \in L^1$$

And

$$\left| \int f \, d\lambda \right| \leq \int |f| \, d\lambda$$

Hint $\quad f = f_+ - f_-, \quad |f| = f_+ + f_-$

② + ③

PROP 3 (HWK)

If f, g are measurable, $g \in L^1$

and $|f| \leq |g|$

Then $f \in L^1$.

PROP 4 LET $f, g \in L^1$, $a, b \in \mathbb{R}$

Then $af + bg \in L^1$ and

$$\int (af + bg)\, d\lambda = a \int f\, d\lambda + b \int g\, d\lambda$$

PF

STS

$$\int af\, d\lambda = a \int f\, d\lambda \qquad ①$$

$$\int (f+g)\, d\lambda = \int f\, d\lambda + \int g\, d\lambda \qquad ②$$

FOR ① IN CASE $a \geq 0$. (You do case $a < 0$)

$$(af)_{\pm} = a f_{\pm}$$

So

$$\int af\, d\lambda = \int af_+\, d\lambda - \int af_-\, d\lambda$$

$$= a \left(\int f_+\, d\lambda - \int f_-\, d\lambda \right) \quad \text{FROM B4}$$

$$= a \int f\, d\lambda$$

FOR ③ Let $h = f + g$

Now $h_+ \le f_+ + g_+$ (You do it)

So $f, g \in L^1 \Rightarrow \int h_+ \, d\lambda \le \int f_+ \, d\lambda + \int g_+ \, d\lambda < \infty$.

Similarly $\int h_- \, d\lambda < \infty$.

So $h \in L^1$.

NEXT $h_+ - h_- = h = f + g$

$$= f_+ - f_- + g_+ - g_-$$

So $h_+ + f_- + g_- = h_- + f_+ + g_+$

By additivity in nonnegative case get

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+$$

$$\int h_+ - \int h_- = \left(\int f_+ - \int f_-\right) + \left(\int g_+ - \int g_-\right)$$

$$\int h = \int f + \int g \qquad \checkmark$$

LEBESGUE DOMINATED CONVERGENCE THM 6 (LDCT) 5

Suppose $\{f_k\}_{k=1}^{\infty}$ is sequence of measurable f^ns on $\mathbb{R}^n$ for which

(a) $f(x) = \lim_{k\to\infty} f_k(x)$ ∃ all $x \in \mathbb{R}^n$

(b) $\exists\, g \in L^1$ (DOMINATING FUNCTION);
$\forall k$
$$|f_k(x)| \leq g(x) \quad \forall x \in \mathbb{R}^n$$

Then $f \in L^1$ and
$$\int f\,d\lambda = \lim_{k\to\infty} \int f_k\,d\lambda$$

ie
$$\int \left(\lim_{k\to\infty} f_k\right) d\lambda = \lim_{k\to\infty} \int f_k\,d\lambda.$$

PF By (b) $|f(x)| \leq g(x)$ holds

So by PROP 3, $f \in L^1$.

Similarly $f_k \in L^1$ by (b)

Let $h_k = g + f_k \geq 0$.

By FATOU'S LEMMA

$$\int (g+f)\,d\lambda \leq \liminf_{k\to\infty} \int (g+f_k)\,d\lambda$$

i.e. $$\int g\,d\lambda + \int f\,d\lambda \leq \cancel{\int g\,d\lambda} + \liminf_{k\to\infty} \int f_k\,d\lambda$$

So

$$\int f\,d\lambda \leq \liminf_{k\to\infty} \int f_k\,d\lambda \quad \text{(A)}$$

Applying same idea to $h_k \equiv g - f_k \geq 0$

And using $\liminf(-f_k) = -\limsup f_k$

we get

$$\limsup_{k\to\infty} \int f_k\,d\lambda \leq \int f\,d\lambda \quad \text{(B)}$$

Claiming (A) + (B) get

$$\limsup_{k\to\infty} \int f_k\,d\lambda \overset{(B)}{\leq} \int f\,d\lambda \overset{(A)}{\leq} \liminf_{k\to\infty} \int f_k\,d\lambda$$
$$\leq \limsup_{k\to\infty} \int f_k\,d\lambda$$

So have = and hence $\lim_{k\to\infty} \int f_k\,d\lambda$ $\exists$ + result ✓ $\square$

REMARKS ON "ALMOST EVERYWHERE" [J, 6.C]

DEF^n 7 Let $f, g : \mathbb{R}^n \to [-\infty, +\infty]$ be measurable

We say $f = g$ a.e. if $\lambda(\{x \in \mathbb{R}^n / f(x) \neq g(x)\}) = 0$

PROP 8 (HWK)

(A) Let $f \geq 0$ be measurable

Then

$$\int f \, d\lambda = 0 \iff f = 0 \text{ a.e.}$$

(B) If f is measurable and $f = 0$ a.e.

Then $f \in L^1$ and $\int f \, d\lambda = 0$

DEF 8 (A) Suppose $f : \mathbb{R}^n \to [-\infty, +\infty]$ is __defined__ only a.e.

We say f is __measurable__ if the function

$$g(x) = \begin{cases} f(x) & \text{if } f \text{ is defined at } x \\ 0 & \text{otherwise} \end{cases}$$

is measurable.

(B) Suppose f is defined a.e. and

(i) • g is defined on all of $\mathbb{R}^n$.

(ii) • g is measurable and $g \in L^1$

(iii) • $f = g$ a.e.

Then we say $f \in L^1$ and __define__ $\int f \, d\lambda = \int g \, d\lambda$

NOTE HWK problem shows that it doesn't matter which function g you pick in B

If $\tilde{g}$ is another function satisfying (i)–(ii)

Then $\int \tilde{g}\, d\lambda = \int g\, d\lambda$.

So $\int f\, d\lambda$ is well-defined

USING DEFS A, B we can extend LDCT to a.e. defined functions

LDCT a.e. [THM 9]

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions that are defined a.e. and are measurable

Suppose also that g is defined a.e. with $g \in L^1$ and

(a) $f(x) = \lim_{k\to\infty} f_k(x)$ $\exists$ a.e.

(b) $|f_k(x)| \leq g(x)$ a.e.

Then

$$\int f(x)\, d\lambda = \lim_{k\to\infty} \int f_k(x)\, d\lambda$$

PROOF

Each statement in hypothesis holds except on an associated null set.

There are a countable # of such statements, which are all simultaneously true except on a "giant" null set that is the union of all individual null sets.

Define all f^ns to be 0 on this giant null set.

Now apply original LDCT

$\square$

INTEGRATION OVER SUBSETS OF $\mathbb{R}^n$ [J, 6.D]

DEF 10

(A) Let $f : \mathbb{R}^n \to [0, +\infty]$ be measurable and let $E \subseteq \mathbb{R}^n$ be measurable.

Since $f \chi_E$ is measurable + nonnegative we can define

$$\int_E f \, d\lambda := \int f \chi_E \, d\lambda$$

(B) Let $f : \mathbb{R}^n \to [-\infty, +\infty]$ be measurable f^n and let $E \subseteq \mathbb{R}^n$ be measurable set

If $f\chi_E \in L^1$ we define

$$\int_E f \, d\lambda := \int (f\chi_E) \, d\lambda$$

and say $f \in L^1(E)$.

(C) Suppose $E \subseteq \mathbb{R}^n$ is a measurable set

Let $f : E \to [-\infty, +\infty]$ be a function

Define

$$g(x) = \begin{cases} f(x) & x \in E \\ 0 & x \in \mathbb{R}^n \setminus E \end{cases}$$

(i) We say f is <u>measurable</u> on E if

g is measurable on $\mathbb{R}^n$

(ii) We say $f \in L^1(E)$ if $g \in L^1(\mathbb{R}^n)$ and define

$$\int_E f \, d\lambda = \int g \, d\lambda$$

THM 11 [COUNTABLE ADDITIVITY OF INTEGRAL]

Suppose $E_1, E_2, \ldots$ is a countable collection of disjoint measurable subsets of $\mathbb{R}^n$ and

$$E = \bigcup_{k=1}^{\infty} E_k.$$

Suppose f is measurable on E and either

(a) $f \geq 0$

or (b) $f \in L^1(E)$

Then

$$\boxed{\int_E f \, d\lambda = \sum_{k=1}^{\infty} \int_{E_k} f \, d\lambda}$$

PROOF

$\boxed{\text{CASE } f \geq 0}$

$$\int_E f d\lambda = \int f \chi_E \, d\lambda$$

$$= \int \sum_{k=1}^{\infty} f \chi_{E_k} \, d\lambda$$

$$\overset{\text{MCT}}{=} \sum_{k=1}^{\infty} \int f \chi_{E_k} \, d\lambda = \sum_{k=1}^{\infty} \int_{E_k} f \, d\lambda.$$

$\boxed{\text{CASE } f \in L^1(E)}$

LET $F_n = \sum_{k=1}^{n} f \chi_{E_k}$

LET'S COMPUTE + JUSTIFY ONLY LATER:

$$
\begin{aligned}
\int_E f \, d\lambda &= \int f \chi_E \, d\lambda \\
&= \int \sum_{k=1}^{\infty} f \chi_{E_k} \, d\lambda \\
&= \int \lim_{n\to\infty} F_n \, d\lambda \\
&\overset{\circledast}{=} \lim_{n\to\infty} \int F_n \, d\lambda \\
&= \lim_{n\to\infty} \int \sum_{k=1}^{n} f \chi_{E_k} \, d\lambda \\
&\underset{\text{ADDITIVITY}}{\overset{\text{FINITE}}{=}} \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f \, d\lambda \\
&= \sum_{k=1}^{\infty} \int_{E_k} f \, d\lambda
\end{aligned}
$$

To JUSTIFY $\circledast$ we need to apply LDCT to F_n.

Well
- $F_n \to \sum_{k=1}^{\infty} f \chi_{E_k}$ ✓
- $|F_n| \le |f|$ ~~where $g := F_1$~~

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as $$|F_n| = \left|\sum_{k=1}^{n} f \chi_{E_k}\right| \overset{\Delta}{\leq} \sum_{k=1}^{n} |f| \chi_{E_k}$$

$$= |f| \sum_{k=1}^{n} \chi_{E_k}$$

$$\leq |f| \quad \text{as } E_k \text{ are } \underline{\underline{\text{DISJOINT}}}.$$

Since $f \in L^1$, $|f| \in L^1$ holds

So LDCT applies and ⊛ is true

∎

THM 12 BOUNDED CONVERGENCE THM

LET $X \subseteq \mathbb{R}^n$ with $\lambda(X) < \infty$.
Suppose $f_k : X \to \mathbb{R}$ be measurable with

$$f_k \to f \quad PW.$$

Suppose $\exists M > 0 : |f_k(x)| \leq M \quad \forall k, x.$
Then

$$\lim_{k \to \infty} \int f_k \, d\lambda = \int f \, d\lambda$$

PROOF Let $g(x) = M$ (Simple)

Since $\lambda(X) < \infty$ $g \in L^1(X)$

Result now follows from LDCT.