INTEGRATION

[J, 6.A, 6.B]

NONNEGATIVE FUNCTIONS

DEF 1 Let $s : \mathbb{R}^n \to [0, \infty)$ be a measurable simple function that is nonnegative and finite.

If

$$\boxed{s = \sum_{k=1}^{m} \alpha_k \chi_{A_k}} \quad (1)$$

where $0 \leq \alpha_k < \infty$ and A_k are measurable and DISJOINT

Then we define

$$\boxed{\int s \, d\lambda = \sum_{k=1}^{m} \alpha_k \lambda(A_k)} \quad (2)$$

CONVENTION : $0 \cdot \infty = 0$.

NOTE : The expansion (1) is NOT unique!!

PROPERTIES

(S1) $\int s \, d\lambda$ is well defined independent of choice of expansion in (1).

(S2) $0 \leq \int s \, d\lambda \leq \infty$

(S3) $\int cs \, d\lambda = c \int s \, d\lambda$ for const $0 \leq c < \infty$

S4 If $s, t : \mathbb{R}^n \to [0, \infty)$ are measurable simple f^{ns}

Then $$\int (s+t)\,d\lambda = \int s\,d\lambda + \int t\,d\lambda$$

S5 If $s, t : \mathbb{R}^n \to [0, \infty)$ are measurable simple f^{ns} with $s \le t$

Then $$\int s\,d\lambda \le \int t\,d\lambda$$

PROOF

S5 Suppose
$$s = \sum_{k=1}^{m} \alpha_k \chi_{A_k}$$
$$t = \sum_{j=1}^{l} \beta_j \chi_{B_j}$$

We can assume $\bigcup_{k=1}^{m} A_k = \mathbb{R}^n = \bigcup_{j=1}^{l} B_j$

as we are allowing α's to be zero.

Then $$\int s\,d\lambda = \sum_{k=1}^{m} \alpha_k \lambda(A_k) = \sum_{k=1}^{m} \alpha_k \lambda\left(A_k \cap \left(\bigcup_{j=1}^{l} B_j\right)\right)$$

$$= \sum_{k=1}^{m} \alpha_k \; \lambda\Big(\underbrace{\bigcup_{j=1}^{l} (A_k \cap B_j)}_{\text{DISJOINT UNION}}\Big)$$

$$= \sum_{k=1}^{m} \sum_{j=1}^{l} \alpha_k \; \lambda(A_k \cap B_j)$$

by additivity of λ.

And

$$\int t \, d\lambda = \sum_{k=1}^{m} \sum_{j=1}^{l} \beta_j \; \lambda(A_k \cap B_j).$$

$\boxed{\text{Fix } j, k}$

(a) If $\lambda(A_k \cap B_j) \neq 0$ $\quad \exists \, x \in A_k \cap B_j$

So $\quad \alpha_k = s(x) \leq t(x) = \beta_j$

So $\quad \alpha_k \lambda(A_k \cap B_j) \leq \beta_j \lambda(A_k \cap B_j) \quad (*)$

(b) If $\lambda(A_k \cap B_j) = 0$ then $(*)$ trivially true.

SUMMING $(*)$ over all j, k gives

$$\int s \, d\lambda \leq \int t \, d\lambda.$$

S1 Suppose

$$s = \sum_{k=1}^{m} \alpha_k \chi_{A_k}$$

$$t = s = \sum_{j=1}^{l} \beta_j \chi_{B_j}$$

are 2 representations of s.

By S5 since $t \le s$ $\quad \int t \, d\lambda \le \int s \, d\lambda$
and $t \ge s$ $\quad \int t \, d\lambda \ge \int s \, d\lambda$

So

$\int t \, d\lambda = \int s \, d\lambda$ is well

defined independent of choice of repn

S2 + S3 : EASY.

S4 With notation as above

$$s+t = \sum_{k=1}^{m} \sum_{j=1}^{l} (\alpha_k + \beta_j) \chi_{A_k \cap B_j}$$

So

$$\int (s+t)\, d\lambda = \sum_{k=1}^{m} \sum_{j=1}^{l} (\alpha_k + \beta_j)\, \lambda(A_k \cap B_j)$$

$$= \int s\, d\lambda + \int t\, d\lambda$$

as above

$\square$

DEF 2 LET $f : \mathbb{R}^n \to [0, \infty]$ be measurable

Define LEBESGUE INTEGRAL OF f to be

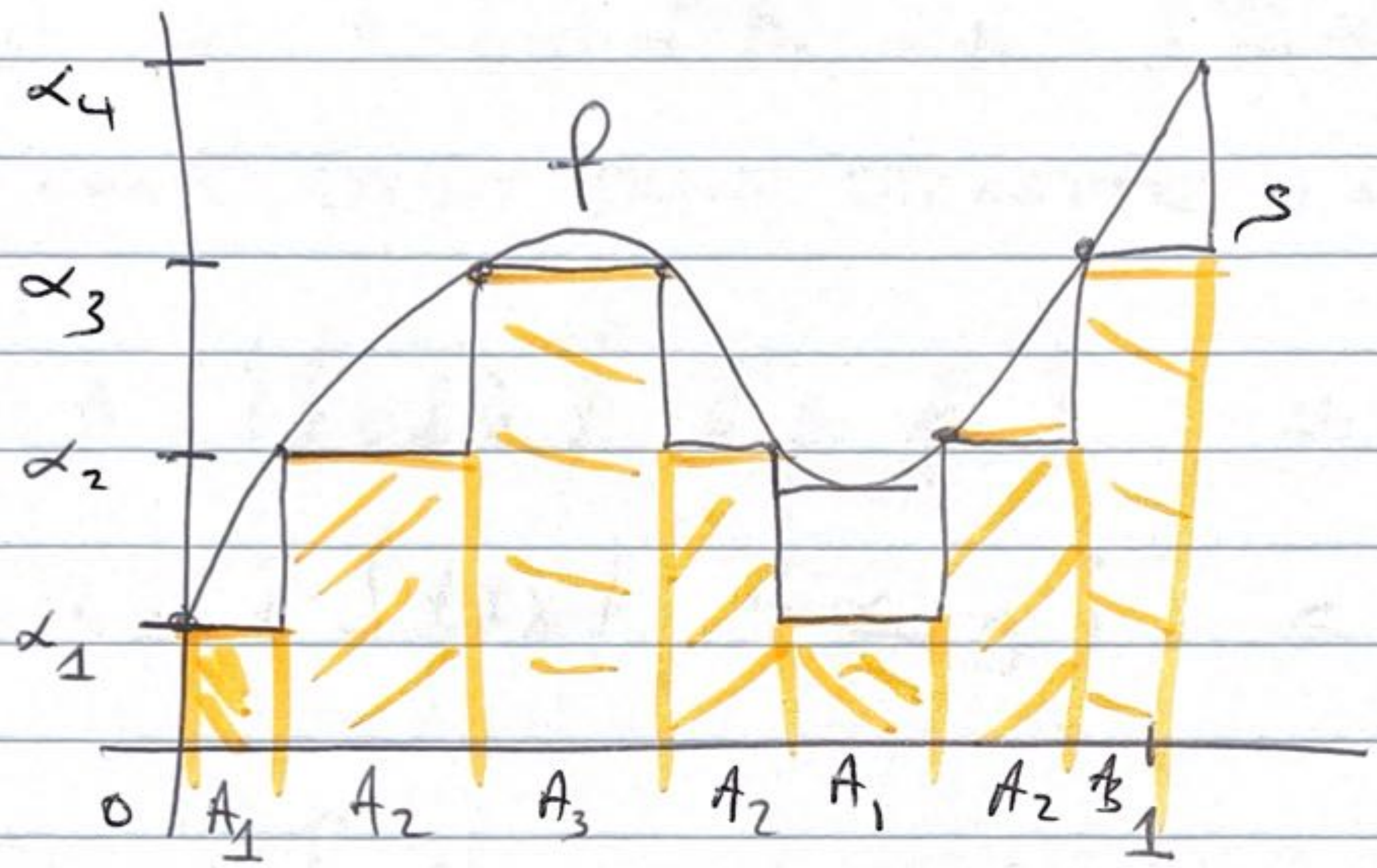
$$\int f\, d\lambda = \sup \Big\{ \int s\, d\lambda \,\Big/\, s \leq f$$

where $s : \mathbb{R}^n \to [0, \infty)$ is simple measurable $\Big\}$

So we approximate $\int f\, d\lambda$ from below by integrals of simple functions

PICTURE

Here is $s \le f$.



NOTE

It is "trivial" to check (S1) (S2) (S3) (S5) hold for $\int f \, d\lambda$.

EG If $f \le g$ Then $\int f \, d\lambda \le \int g \, d\lambda$

SINCE

If $s \le f$ Then $s \le g$

So $\{ \int s \, d\lambda \mid s \le f \} \le \{ \int s \, d\lambda \mid s \le g \}$

So $\int f \, d\lambda = \sup \{ \quad \} \le \sup \{ \} = \int g \, d\lambda$

WHAT ABOUT S4?

CLAIM If $f, g : \mathbb{R}^n \to [0, \infty]$ are measurable

Then

$$\int f \, d\lambda + \int g \, d\lambda \leq \int (f+g) \, d\lambda$$

BUT CAN'T EASILY PROVE OPPOSITE INEQUALITY!!

PF

Let $A = \left\{ \int s \, d\lambda \;/\; s \leq f \right\} \subseteq \mathbb{R}$

$B = \left\{ \int t \, d\lambda \;/\; t \leq g \right\} \subseteq \mathbb{R}$

Now

$$\begin{aligned}
\int f \, d\lambda + \int g \, d\lambda &= \sup(A) + \sup(B) \\
&\overset{*}{\leq} \sup(A+B) \\
&= \sup \left\{ \int s \, d\lambda + \int t \, d\lambda \;/\; s \leq f, t \leq g \right\} \\
&\cancel{\leq \sup \left\{ \int s \, d\lambda + \int \right.} \\
&\leq \sup \underbrace{\left\{ \int (s+t) \, d\lambda \;/\; s+t \leq f+g \right\}}_{C} \\
&\leq \int (f+g) \, d\lambda
\end{aligned}$$

as $A+B \subseteq C$ as $s \leq f, t \leq g \Rightarrow s+t \leq f+g$

PROOF of 7

LEMMA Let $A, B \subseteq \mathbb{R}$. Then

$$\sup(A) + \sup(B) \leq \sup(A+B).$$

Pf

Let $\varepsilon > 0$. $\exists x_* \in A$: $\sup(A) - \varepsilon \leq x_*$
$\exists y_* \in B$ $\sup(B) - \varepsilon \leq y_*$

So

$$\sup(A) + \sup(B) - 2\varepsilon \leq x_* + y_* \leq \sup(A+B)$$

as $x_* + y_* \in A+B$.

The proof that

$$\int (f+g)\, d\lambda = \int f\, d\lambda + \int g\, d\lambda$$

is much harder and relies on The Monotone Convergence Theorem.

~~This result follows from~~

MONOTONE CONVERGENCE THM 4

Suppose $f_k : \mathbb{R}^n \to [0, \infty]$ are measurable functions with

$$0 \le f_1 \le f_2 \le f_3 \le \ldots$$

Let $f(x) = \lim_{k\to\infty} f_k(x)$

Then

$$\lim_{k\to\infty} \int f_k \, d\lambda = \int f \, d\lambda$$
$$= \int \lim_{k\to\infty} f_k \, d\lambda$$

NOTES

① If $0 \le f_1(x) \le f_2(x) \le \ldots$
Then
$$f(x) = \lim_{k\to\infty} f_k(x) = \sup \{ f_k(x) / k = 1, 2, 3 \ldots \}$$
EXISTS.

② Since $f_k \le f_{k+1}$, $\int f_k \le \int f_{k+1}$ $\infty \uparrow$.
So $\lim_{k\to\infty} \int f_k$ $\exists$ ~~too~~.

Assuming MCT is true we can prove 3 results

PROP 5

Let $f, g : \mathbb{R}^n \to [0, \infty]$ be measurable
Then
$$\int (f+g)\, d\lambda = \int f\, d\lambda + \int g\, d\lambda$$

PF As in

$\exists$ a sequence $s_k \to f$ with $0 \leq s_1 \leq s_2 \leq \ldots$
and each s_k a simple, measurable f^n

Similarly $\exists$ $t_k \to f$ $0 \leq t_1 \leq t_2 \leq \ldots$

Then $u_k \equiv s_k + t_k$ ~~has~~ are simple measurable f^{ns}
with
$$0 \leq u_1 \leq u_2 \leq \ldots \qquad \text{and } u_k \to f+g$$

So by MCT x2

$$\int (f+g)\, d\lambda \overset{MCT}{=} \lim_{k\to\infty} \int u_k \, d\lambda$$
$$= \lim_{k\to\infty} \int (s_k + t_k)\, d\lambda$$

$$= \lim_{k\to\infty} \left[\int s_k \, d\lambda + \int t_k \, d\lambda \right]$$

by (S4) as s_k, t_k are simple

$$= \lim_{k\to\infty} \int s_k \, d\lambda + \lim_{k\to\infty} \int t_k \, d\lambda$$

$$\overset{MCT}{=} \int f \, d\lambda + \int g \, d\lambda$$

[]

as $s_k \le s_{k+1} \to f$.

THM 6

Let $f_k \geq 0$ be measurable f^{ns} on $\mathbb{R}^n$

Then

$$\int \left(\sum_{k=1}^{\infty} f_k \right) d\lambda = \sum_{k=1}^{\infty} \int f_k \, d\lambda$$

PF Let $F_n = \sum_{k=1}^{n} f_k$ $\quad F = \sum_{k=1}^{\infty} f_k = \lim_{n\to\infty} F_n$.

Each F_n is measurable and

$$\cdots \le F_n \le F_{n+1} \le \cdots \quad \text{as } f_k \geq 0 \ \forall k$$

So by MCT

$$\int \sum_{k=1}^{\infty} f_k \, d\lambda = \int F \, d\lambda = \lim_{n\to\infty} \int F_n \, d\lambda$$

$$= \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k \, d\lambda$$

$$\overset{\text{PROP 5}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \, d\lambda \quad !!$$

$$= \sum_{k=1}^{\infty} \int f_k \, d\lambda$$

$\square$

FATOU'S LEMMA 7

Let $f_k : \mathbb{R}^n \to [0, \infty]$ be measurable f^{ns}.
Then

$$\int \left(\liminf_{k\to\infty} f_k\right) d\lambda \leq \liminf_{k\to\infty} \int f_k \, d\lambda$$

PROOF

RECALL $\left(\liminf_{k\to\infty} f_k\right)(x) = \liminf_{k\to\infty} f_k(x)$

$$= \lim_{k\to\infty} \left[\inf_{l \geq k} f_l(x) \right]$$

$$= \lim_{k\to\infty} g_k(x)$$

where

$$g_k(x) = \inf_{l \geq k} f_l(x) \quad \text{are measurable.}$$
$$= \inf \{ f_k(x), f_{k+1}(x), \dots \}$$

NOTICE

- $0 \leq g_k(x) \leq f_k(x)$ as $g_k(x)$ is BLB.
- $0 \leq g_k(x) \leq g_{k+1}(x) \ \forall k$ as taking INF over smaller set.

So can apply MCT to g_k:

$$\int (\liminf_{k \to \infty} f_k) \, d\lambda = \int (\lim_{k \to \infty} g_k) \, d\lambda$$
$$\overset{MCT}{=} \lim_{k \to \infty} \int g_k \, d\lambda$$
$$= \liminf_{k \to \infty} \int g_k \, d\lambda$$
$$\leq \liminf_{k \to \infty} \int f_k \, d\lambda$$

as $g_k \leq f_k \Rightarrow \int g_k \, d\lambda \leq \int f_k \, d\lambda$.

$\square$

PROOF OF MCT

We have $0 \le f_1 \le f_2 \le f_3 \le \dots \le f = \lim_{k\to\infty} f_k$

all measurable.

We must show

$$I := \lim_{k\to\infty} \int f_k \, d\lambda = \int f \, d\lambda.$$

(A) SINCE $f_k \le f_{k+1} \le f$

$$\int f_k d\lambda \le \int f_{k+1} d\lambda \le \int f d\lambda$$

and $\cancel{I \le \int f}$

$$I = \underbrace{\sup_k}_{\text{LUB}} \int f_k \, d\lambda \le \underbrace{\int f d\lambda}_{\text{UB}}$$

(B) So must show $I \ge \int f d\lambda$

Let $c < \int f d\lambda$ be arbitrary.

Must show $\boxed{I \ge c}$

Since $\int f d\lambda = \sup\{\int s d\lambda \,/\, s \leq f\}$

We know from def^n sup $\exists$ simple $s \leq f$

So That

$$c < \int s d\lambda.$$

Write

$$s = \sum_{i=1}^{N} \alpha_i \chi_{A_i} \qquad 0 < \alpha_i < \infty$$

A_i DISJT, m'ble

By slightly decreasing α_i's we can assume

$\exists$ simple, measurable s so That

(a) $0 \leq s \leq f$

(b) If $x \in \mathbb{R}^n$ is such That $f(x) > 0$

Then $s(x) < f(x)$

(c) $\int s d\lambda > c.$

(C) Now let

$$E_k = \{x \in \mathbb{R}^n \,/\, f_k(x) \geq s(x)\}$$

Since $f_k - s$ is measurable f^n, E_k is measurable set.

Since $f_{k+1} \geq f_k$ we have

$$E_1 \subset E_2 \subset E_3 \subset \dots$$

and CLAIM $\bigcup_{k=1}^{\infty} E_k = \mathbb{R}^n$.

PF OF CLAIM

Let $x \in \mathbb{R}^n$.

- If $f(x) = 0$ Then $s(x) = 0$ by ⓐ

  and $f_k(x) = 0 \quad \forall k$

  So $x \in E_k \quad \forall k$.

- If $f(x) > 0$ Then by ⓑ $f(x) > s(x)$

  So $\exists K : \forall k \geq K \quad f_k(x) > s(x)$

  as $f_k \to f$ PW.

  So $x \in E_k$.

- UPSHOT $x \in \bigcup_{k=1}^{\infty} E_k$. $\square$

NOW
$$f_k \geq f_k \chi_{E_k}$$
$$\geq s\,\chi_{E_k} \qquad \text{by def}^n\ E_k$$
$$= \sum_{i=1}^{N} \alpha_i \chi_{A_i \cap E_k}$$

So
$$\int f_k\, d\lambda \geq \sum_{i=1}^{N} \alpha_i\, \lambda(A_i \cap E_k)$$

Recall

MS If $B_1 \subset B_2 \subset B_3 \subset \ldots$

Then
$$\lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \lambda(B_k)$$

Apply MS to $B_k = A_i \cap E_k$, using $\bigcup_{k=1}^{\infty} B_k = A_i$ by CLAIM.

So
$$\lim_{k\to\infty} \lambda(A_i \cap E_k) = \lambda(A_i)$$

$\therefore$
$$\int f_k\, d\lambda \geq \sum_{i=1}^{N} \alpha_i\, \lambda(A_i) = \int s\, d\lambda > c$$

So
$$I = \lim_{k\to\infty} \int f_k\, d\lambda \geq c$$

$\square$