LUSIN'S THM

[A, 2E], [HEL] ①

LUSIN'S THM VERSION A (THM 1)

Let $X \subseteq \mathbb{R}^n$ be measurable and $f : X \to \mathbb{R}$ be measurable.

Then $\forall \varepsilon > 0$ $\exists$ closed $F \subseteq X$ so that

ⓐ $\lambda(X \sim F) < \varepsilon$

ⓑ $f|_F$ is CTS in that

$\forall x_k, x \in F$ with $x_k \to x$

we have $f(x_k) \to f(x)$

NOTE

$f|_F$ CTS is not same as $f : X \to \mathbb{R}$ is CTS

at each point of F. since f CTS at $x \in F$

means $\boxed{\forall x_k \in X}$ with $x_k \to x$ we have

$f(x_k) \to f(x)$

EX 2 LET $f : [0,1] \to \mathbb{R}$ be the DIRICHLET FUNCTION

$$f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

NOTICE $f|_{[0,1] \sim \mathbb{Q}} \equiv 0$ is CTS !!

However $[0,1] \sim \mathbb{Q}$ is not closed

LET $\varepsilon > 0$

We can construct F as follows

Let $\{r_n\}_{n=1}^{\infty}$ enumerate $\mathbb{Q} \cap [0,1]$.

Let $G_n = \left(r_n - \frac{\varepsilon}{2^{n+1}}, r_n + \frac{\varepsilon}{2^{n+1}} \right)$ Open

Then $G = \bigcup_{n=1}^{\infty} G_n$ is open and G contains $\mathbb{Q} \cap [0,1]$.

Also $\lambda(G) \leq \sum_{n=1}^{\infty} \lambda(G_n) = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^{n}} = \varepsilon$

So $F := [0,1] \sim G$ is Closed

and $\lambda(F) < \varepsilon$,

Since $F \cap \mathbb{Q} = \emptyset$ we have $f|_F \equiv 0$ is CTS.

PROOF OF LUSIN'S THM

For simplicity we assume ~~...~~ set X is BOUNDED.

$\boxed{\text{CASE } f \text{ IS SIMPLE}}$

Suppose $\phi = \sum_{j=1}^{n} a_j \chi_{A_j}$ is simple,

where the A_j are disjoint m'ble sets. with ~~$A_1 \cup \dots A_n = X$~~. LET $A_0 = X \sim (A_1 \cup \dots \cup A_n)$, A_0 IS M'BLE.

LET $\varepsilon > 0$. By The Approx^n Thm $\exists$ $\underset{\text{CLOSED}}{F_j} \subset A_j \subset \underset{\text{OPEN}}{G_j}$

with $\lambda(G_j \sim F_j) < \frac{\varepsilon}{n+1}$, $j = 0, \dots, n$.

Since $A_j \sim F_j \subset G_j \sim F_j$ we also have

$$\lambda(A_j \sim F_j) < \frac{\varepsilon}{n+1}$$

Let

$$F = \bigcup_{j=0}^{n} F_j \qquad \underline{\text{CLOSED}}$$

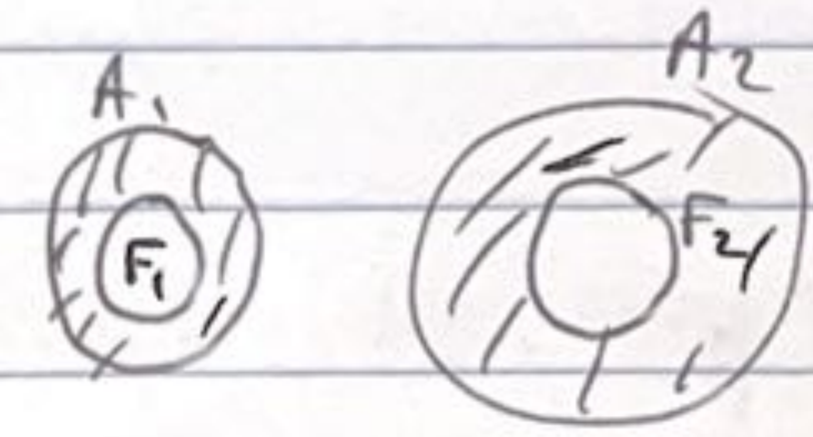
and

$$X \sim F = (A_0 \cup \dots \cup A_n) \sim (F_0 \cup \dots \cup F_n)$$

$A_1 \cap A_2 = \phi$

$$= (A_0 \sim F_0) \cup \dots \cup (A_n \sim F_n)$$

DISJOINT UNION.



So

$$\lambda(X \sim F) = \sum_{j=0}^{n} \lambda(A_j \sim F_j) < \sum_{j=0}^{n} \frac{\varepsilon}{n+1} = \varepsilon$$

Since X is bounded, the closed sets $F_j \subset X$ are compact

Since $F_j \cap F_k = \emptyset$ we know

$$\text{dist}\,(F_j, F_k) > 0 \qquad \forall\, j \neq k$$

Since ϕ is constant on each A_j

ϕ ———————— F_j

$\therefore\ \phi|_F$ is CONTINUOUS.

Here we use:

<u>LEMMA 3</u>

Suppose F_1, F_2 are closed sets with $\text{dist}(F_1, F_2) > 0$ and $\phi|_{F_j}$ is CTS for $j = 1, 2$.

Then $\phi : F_1 \cup F_2 \to \mathbb{R}$ is also CTS.

<u>PROOF IDEA</u> LET $x \in F_1 \cup F_2$. WLG $x \in F_1$

If $\{x_k\}_{k=1}^{\infty} \subseteq F_1 \cup F_2$ with $x_k \to x$

Then $\exists N$; $\forall k \geq N$ $x_k \in F_1$.

Since $\phi|_{F_1}$ is CTS, $\phi(x_k) \to \phi(x)$ as req'd.

$\square$

$\boxed{\text{CASE ARBITRARY MBLE } f}$ 5

Now let $f : X \to \mathbb{R}$ be measurable.

Then $\exists$ Simple $\phi_n \to f$ P.W. $n = 1, 2, 3, \ldots$

LET $\varepsilon > 0$

By previous case $\exists$ Closed $F_n \subseteq X$:

$$\text{a} \quad \lambda(X \sim F_n) < \frac{\varepsilon}{2^{n+1}}$$

$$\text{b} \quad \phi_n|_{F_n} \text{ IS CT}$$

By EGOROV'S THM 9 since $\lambda(X) < \infty$

$\exists$ Closed $F_0 \subseteq X$:

$$\text{c} \quad \lambda(X \sim F_0) < \varepsilon/2$$

$$\text{d} \quad \phi_n \to f \text{ UNIFORMLY ON } F_0$$

LET

$$F = \bigcap_{n=0}^{\infty} F_n.$$

U.B.V. F is CLOSED.

Also $\cancel{\lambda(F)}$ $\lambda(X \sim F) = \lambda\left(X \sim \bigcap_{n=0}^{\infty} F_n\right)$

$$= \lambda\left(\bigcup_{n=0}^{\infty}(X \sim F_n)\right) \leq \sum_{n=0}^{\infty} \lambda(X \sim F_n) = \varepsilon \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \varepsilon$$

Since $F_n \subset F$ and $\phi_n|_{F_n}$ is CTS, $\phi_n|_F$ is CTS.

Since $\phi_n \to f$ UNIF on F

$f|_F$ IS ALSO CTS

$\square$

LUSIN'S THM VERSION B (THM 4)

LET $f : \mathbb{R} \to \mathbb{R}$ be measurable.

Then $\forall \varepsilon > 0$ $\exists$ Closed $F \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ CTS with

ⓐ $\lambda(\mathbb{R} \sim F) < \varepsilon$

ⓑ $f|_F = g|_F$

UPSHOT Can modify values of a m'ble f^n on a set of small Lebesgue measure to produce a CTS f^n.

PF LET $\varepsilon > 0$ By LUSIN VERSION A $\exists$ Closed F :

ⓐ $\lambda(\mathbb{R} \sim F) < \varepsilon$

ⓑ $f|_F$ IS CTS.

The result now follows from

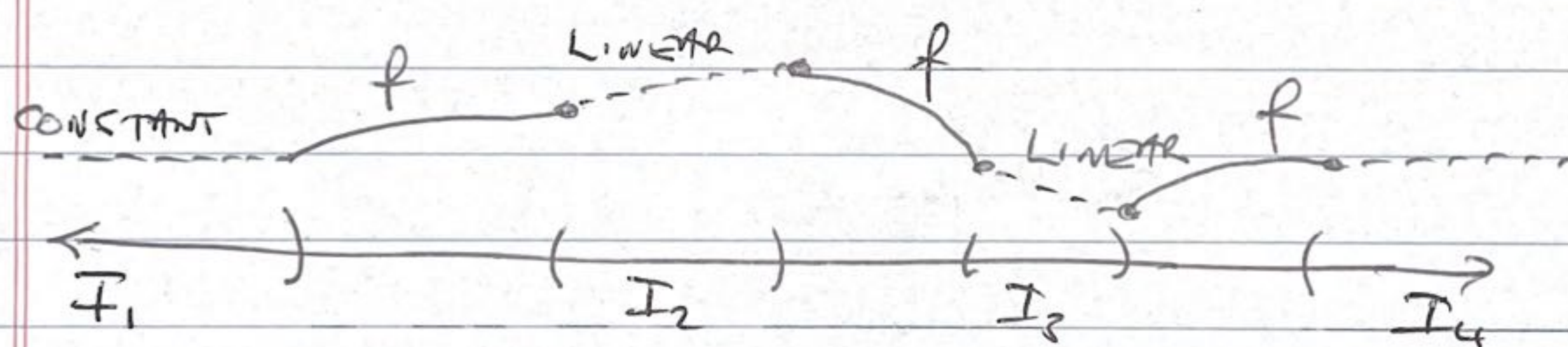
<u>CLAIM</u> Let $F \subseteq \mathbb{R}$ be closed and $f : F \to \mathbb{R}$ CTS.

Then $\exists$ CTS $g : \mathbb{R} \to \mathbb{R}$ with $g|_F = f$

<u>PF</u>

Since $\mathbb{R} \sim F$ is open

$$\mathbb{R} \sim F = \bigcup_{k=1}^{\infty} I_k \qquad I_k \text{ disjt open intervals}$$



$\square$