CONVERGENCE IN MEASURE

[HEIL, 3.8]

PROBABILITY THEORY QUESTION

$\exists$? small probability that a random variable X_n differs from a random variable X by more than ε?

LET f_n = Probability Density F^n of X_n
f = ――――――――――――― X.

This becomes the question

Does $f_n \to f$ in measure?

DEF 18 Let $X \subseteq \mathbb{R}^N$
Let $f_n, f : X \to [-\infty, \infty]$ be measurable f^{ns}

that are finite a.e.
We say $f_n \xrightarrow{m} f$ (f_n converges to f in measure) if

$$\forall \varepsilon > 0 \quad \lim_{n \to \infty} \lambda\left(\{x \in X \mid |f(x) - f_n(x)| \geq \varepsilon\}\right) = 0$$

IDEA As $n \to \infty$
$f_n(x)$ gets closer to $f(x)$ except possibly on a set of smaller + smaller measure.

IN COMPLETE DETAIL:

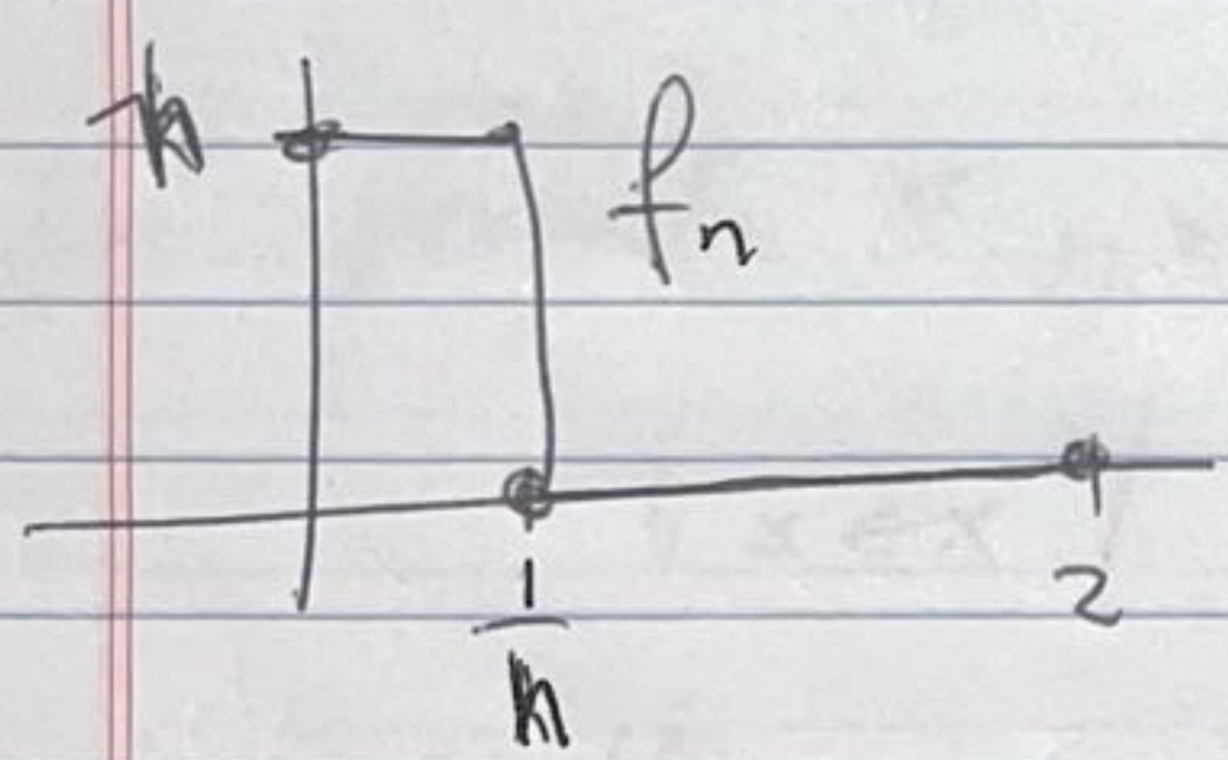
$$\forall \varepsilon, \eta > 0 \quad \exists N : \quad \forall n > N$$

$$\lambda(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) < \eta$$

<u>NEXT</u>

- Examples
- How is convergence in measure related to PW, PW a.e., L^∞ CONVERGENCE?

<u>EX 19</u> $X = [0,2]$



<u>CLAIM</u> $f_n \xrightarrow{m} 0$

LET $\varepsilon < 1$.

Then $\lambda(\{x \in [0,2] \mid |f_n(x)| > \varepsilon\})$

$$= \lambda([0, \tfrac{1}{n}]) = \frac{1}{n}$$

So $\lim\limits_{n\to\infty} \lambda(\{x \mid |f_n(x) - \underset{\overset{\shortparallel}{0}}{f(x)}| > \varepsilon\}) = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

$\square$

PROP 20

Suppose $\lambda(X) < \infty$ and $f_n \to f$ PW a.e. with f finite

Then $f_n \xrightarrow{m} f$

PROOF LET $\eta > 0$

By Egorov's Thm $\exists$ Closed $F \subset X$:

(a) $\lambda(X \sim F) < \eta$

(b) $f_n \to f$ UNIFORMLY on F.

(b) means : $\forall \varepsilon > 0 \ \exists N : \forall x \in F, \ \forall n \geq N$

$$|f_n(x) - f(x)| < \varepsilon.$$

So for ~~fixed~~ each $\varepsilon, \eta > 0 \ \exists N = N(\varepsilon, \eta) : \forall n \geq N$

$$\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\} \subset X \sim F$$

So $$\lambda(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) \leq \lambda(X \sim F) < \eta$$

ie $f_n \xrightarrow{m} f$

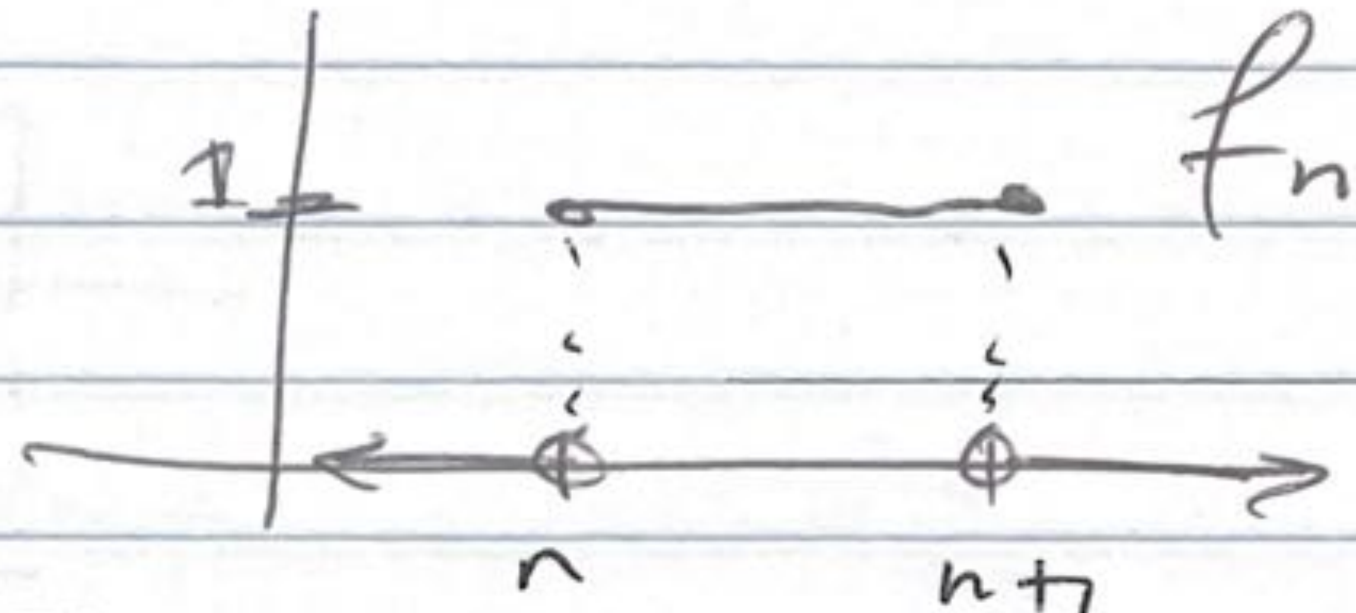
$\square$

EX 21 MOVING BOXES

This example shows that if $\lambda(X) = \infty$ Then

$$f_n \to f \text{ PW} \not\Rightarrow f_n \xrightarrow{m} f$$

LET $f_n = \chi_{[n, n+1]}$



ⓐ $f_n \to 0$ PW as $\boxed{\text{Fix } x>0}$ $\forall \varepsilon > 0$ Choose $N > x$

Then $\forall n \geq N$ $\quad f_n(x) = \chi_{[n, n+1]}(x) = 0$ as $x < n$

So $|f_n(x) - 0| = 0 < \varepsilon$

BUT

ⓑ LET $0 < \varepsilon < 1$

Then

$$\{x \in \mathbb{R} \,/\, |f_n(x) - 0| \geq \varepsilon\} = [n, n+1]$$

So $\lambda(\{x \in \mathbb{R} \,/\, |f_n(x) - 0| \geq \varepsilon\}) = 1 \not\to 0$ as $n \to \infty$

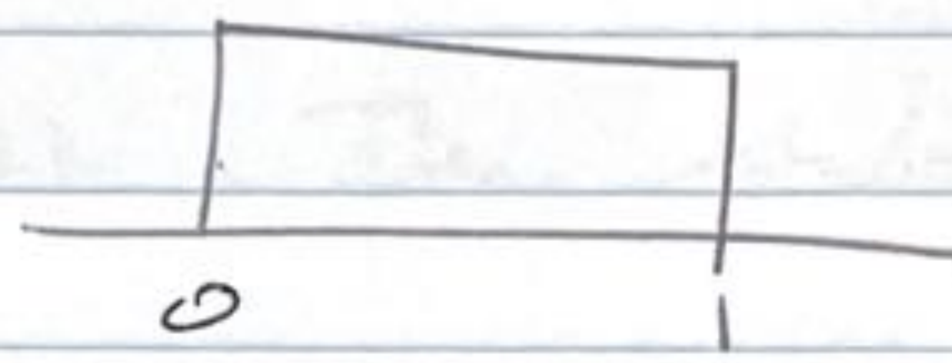
EX 22 BOXES MARCHING IN CIRCLES.

This example shows

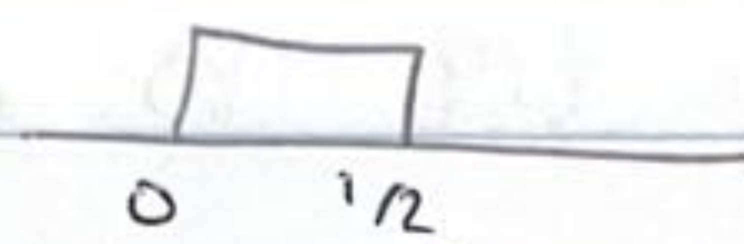
$$f_n \xrightarrow{m} f \not\Rightarrow f_n \to f \text{ PW a.e.}$$

Define

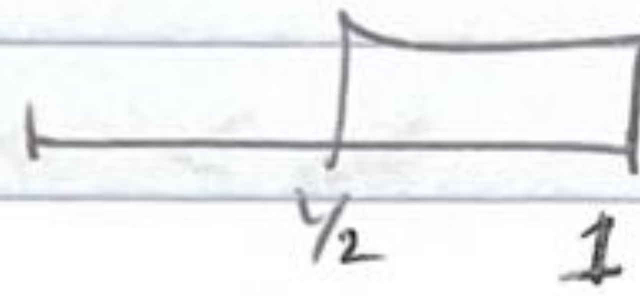
$f_1 = \chi_{[0,1]}$



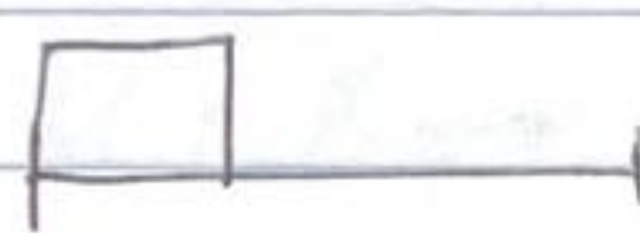
$f_2 = \chi_{[0,\frac{1}{2}]}$



$f_3 = \chi_{[\frac{1}{2},1]}$

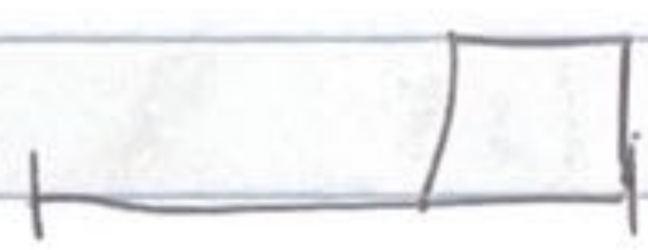


$f_4 = \chi_{[0,\frac{1}{3}]}$



$f_5 = \chi_{[1/3,\, 2/3]}$

$f_6 = \chi_{[2/3,\, 1]}$



NOW LET $0 < \varepsilon < 1$

$$a_n = \lambda(\{|f_n| \geq \varepsilon\}) = \lambda\{\text{SUPPORT OF } f_n\}$$

$$a_n = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$$

So $a_n \to 0$ as $n \to \infty$.

So

$$f_n \xrightarrow{m} 0.$$

BUT $f_n \not\to 0$ PW a.e. $\forall x$ $\exists\, \infty$ many n with $f_n(x) = 1$.

SUMMARY

L^∞ CONVERGENCE $\Longrightarrow$ PW a.e. CONVERGENCE $\overset{\lambda(X)<\infty}{\Longrightarrow}$ CONV. IN MEASURE.

Other implications are not true in general.

NOTICE In EX22 The subsequence $f_k = \chi_{[0,\frac{1}{k}]}$ has $f_k \to 0$ PW a.e. This is no accident:

THM 23 Let $\lambda(X) < \infty$

Suppose $f_n, f : X \to \mathbb{R}$ are measurable with $f_n \xrightarrow{m} f$. Then $\exists$ subsequence $(f_{n_k})_{k=1}^{\infty}$ with $f_{n_k} \to f$ PW a.e.

PROOF

$f_n \xrightarrow{m} f$ means

$\forall \varepsilon, \eta > 0 \quad \exists N = N(\varepsilon, \eta) : \forall n \geq N$

$$\lambda(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) < \eta$$

SET $\varepsilon = \frac{1}{k}$, $\eta = \frac{1}{2^k}$, $n_k = N(\frac{1}{k}, \frac{1}{2^k})$

So $\forall k$ $\exists n_k$:

$$\lambda\{|f_{n_k} - f| > \tfrac{1}{k}\} \le 2^{-k}.$$

LET $B_k = \{|f_{n_k} - f| > \frac{1}{k}\}$ $\quad \lambda(B_k) \le \frac{1}{2^k}$

$$Z = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B_k = \text{“} \limsup_{k\to\infty} B_k \text{”}$$

U BC ✓ $\quad \lambda(Z) = 0$ $\quad$ (Need $\lambda(X) < \infty$ here)

Suppose $x \notin Z$. Then $\exists m$: $\forall k \ge m$

$$|f_{n_k}(x) - f(x)| \le \tfrac{1}{k}$$

So $\quad f_{n_k}(x) \to f(x)$ as $k \to \infty$ for $x \notin Z$

ie $\quad f_n \to f$ PW a.e.

DEF 24 $\quad f_n$ is CAUCHY IN MEASURE if

$\forall \varepsilon, \eta > 0$ $\exists N$: $\forall j, k > N$

$$\lambda(\{|f_j - f_k| \ge \varepsilon\}) < \eta$$

THM 25

Let $f_n : X \to \mathbb{R}$ be measurable f^{ns}.
Then

f_n IS CAUCHY IN MEASURE $\Leftrightarrow$ $\exists$ M'ble $f : f_n \xrightarrow{m} f$.

PF

$\boxed{\Leftarrow}$ Suppose $f_n \xrightarrow{m} f$.

Let $\varepsilon, \eta > 0$. $\exists N : \forall k > N$

$$\lambda(\{|f_k - f| \geq \varepsilon\}) < \eta$$

By Δ Ineq

$$|f_j - f_k| \leq |f_j - f| + |f - f_k| \quad \oplus$$

CLAIM

$$\{|f_j - f_k| \geq 2\varepsilon\} \subseteq \{|f_j - f| \geq \varepsilon\} \cup \{|f_k - f| \geq \varepsilon\}$$

PF

IF $|f_j - f_k| \geq 2\varepsilon$ Then by $\oplus$

$$2\varepsilon \leq |f_j - f| + |f - f_k|$$

Suppose $|f_j - f| < \varepsilon$ ($x \notin$ 1ST SET ON RHS)
Then $\cancel{2\varepsilon \leq}\ |f_k - f| \geq 2\varepsilon - |f_j - f| > \varepsilon$

So $x \in$ 2ND SET ON RHS. $\square$

By Claim

$$\lambda\{|f_j - f_k| \geq 2\varepsilon\} \leq \lambda\{|f_j - f| \geq \varepsilon\} + \lambda\{|f_k - f| \geq \varepsilon\}$$

$$\leq \eta + \eta = 2\eta$$

So f_k is Cauchy in measure ✓

SUPPOSE f_n CAUCHY IN MEASURE (HARD)

$\boxed{\Leftarrow}$ As in 1st part of pf of Thm 23.

$\forall k \; \exists n_k$:

$$\lambda\{|f_{n_{k+1}} - f_{n_k}| \geq 2^{-k}\} < 2^{-k}$$

$$\lambda(B_k) < 2^{-k}.$$

$$Z = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B_k \quad \text{has } \lambda(Z) = 0$$

If $x \notin Z$ can show

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k} \quad \forall k > m.$$

CLAIM $\{f_{n_k}(x)\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$.

For simplicity of notation let

$$g_k := f_{n_k} \quad , \quad H_m = \bigcup_{k=m}^{\infty} B_k \qquad \lambda(H_m) \le 2^{-m+1}$$

Define

$$f(x) = \begin{cases} \lim_{k\to\infty} g_k(x) & x \notin Z \\ 0 & x \in Z \end{cases}$$

SHOW $g_k \xrightarrow{m} f$

Fix $\varepsilon > 0$. choose m: $\frac{1}{2^m} \le \varepsilon$

Pick $x \notin H_m$. IF $n > k > m$ Then

$$|g_n(x) - g_k(x)| \le \sum_{i=k}^{n-1} |g_{i+1}(x) - g_i(x)| \qquad \text{Telescope}$$

$$\le \sum_{i=k}^{n-1} 2^{-i}$$

$$\le 2^{-k+1} \le 2^{-m} \le \varepsilon .$$

Taking $\lim_{n\to\infty} g_n(x) = f(x)$ gives

$\forall x \notin H_m, k > m \ |f(x) - g_k(x)| \le \varepsilon$

So $\{ |f - g_k| \ge \varepsilon \} \subseteq H_m$

$\forall m$, So $\forall k \geq m$ $\quad \lambda\{|f - g_k| \geq \varepsilon\} \leq \lambda(H_m) \leq \frac{1}{2^{m+1}}$

So $\quad g_k \xrightarrow{m} f$.

<u>CLAIM</u>
If f_n Cauchy in Measure and $\exists$ Subsequence

$f_{n_k} \xrightarrow{m} f$ Then $f_n \xrightarrow{m} f$.

<u>PF</u> HWK

$\square$