[A;2E, J:6C, J:10D]
[WZ 4.4]

[HEIL, 3.8, 3.9]

TYPES OF CONVERGENCE

Suppose $f_n : X \to \mathbb{R}$ where $X \subseteq \mathbb{R}^N$.

. We will discuss 5 different notions of convergence of

$$f_n \to f$$

and how they are related.

(I) POINTWISE CONVERGENCE

DEF 1 $f_n \to f$ POINTWISE (PW) on X if $\forall x \in X$

$$f_n(x) \to f(x) \quad \text{as } n \to \infty.$$

ie $\forall x \in X, \forall \varepsilon > 0 \quad \exists N = N(\varepsilon, x) : \quad \forall n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

(II) UNIFORM CONVERGENCE

DEF 2 $f_n \to f$ UNIFORMLY on X if

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : \forall x \in X, \ \forall n \geq N$$

$$|f_n(x) - f(x)| < \varepsilon.$$

NOTE $f_n \to f$ UNIF $\Rightarrow f_n \to f$ PW.

ALT DEF 3

(a) LET $g : X \to \mathbb{R}$. Define

$$\|g\| = \sup_{x \in X} |g(x)|$$

(b) $f_n \to f$ UNIFORMLY $\iff$ $\|f_n - f\| \to 0$ as $n \to \infty$

PF $\boxed{\Leftarrow}$ Suppose $\|f_n - f\| \to 0$ as $n \to \infty$.

Then $\underline{\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \; \forall n \geq N}$ $\|f_n - f\| < \varepsilon$.

So $\underline{\forall x \in X}$

$$\underline{|f_n(x) - f(x)|} \leq \|f_n - f\| \underline{< \varepsilon}$$

So $f_n \to f$ UNIFORMLY

$\boxed{\Rightarrow}$ If $f_n \to f$ UNIFORMLY Then

$\underline{\forall \varepsilon > 0 \quad \exists N = N(\varepsilon)}$: $\forall x \in X$ $\underline{\forall n \geq N}$

$$|f_n(x) - f(x)| < \varepsilon$$

So ε is U.B. for $\{ |f_n(x) - f(x)| \,/\, x \in X \}$

So $\underbrace{\|f_n - f\|}_{\text{LUB}} < \varepsilon$ holds

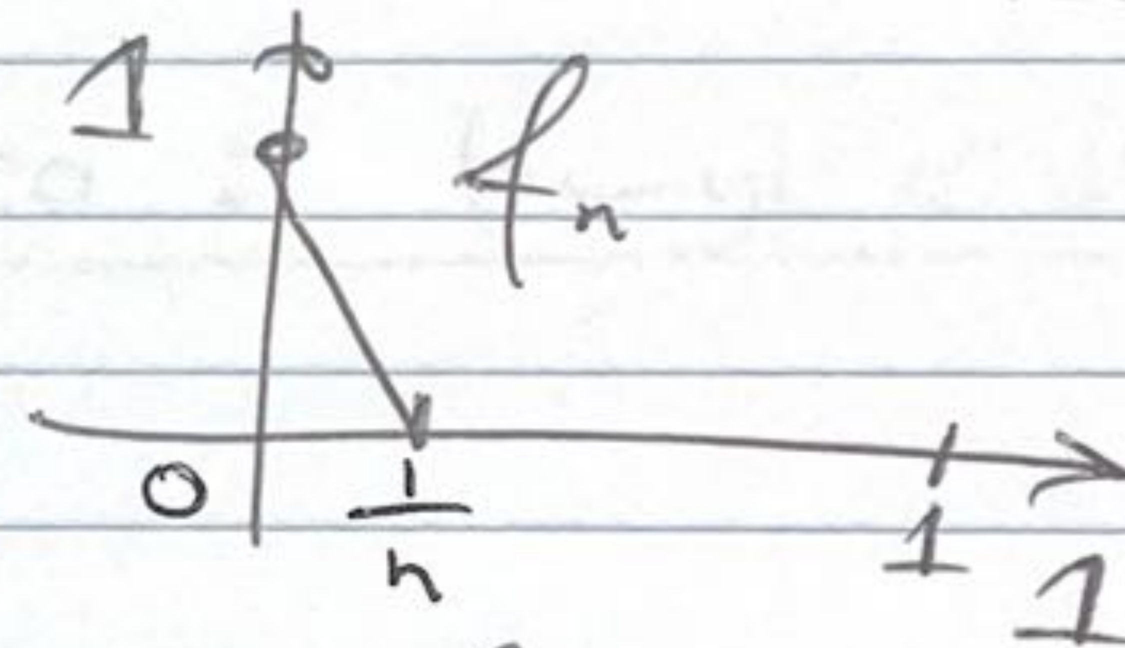
So $\|f_n - f\| \to 0$.

CEX 4 $f_n \to f$ PW $\not\Rightarrow$ $f_n \to f$ UNIFORMLY

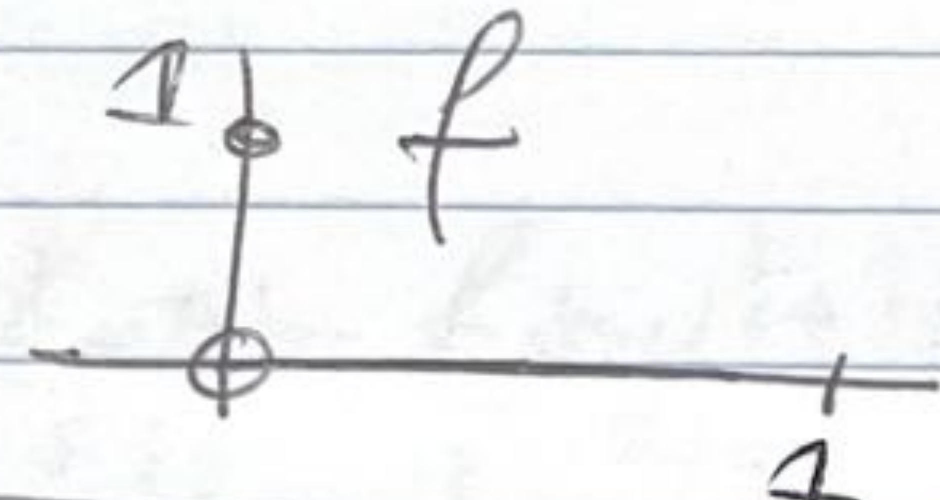
$X = [0,1]$

Since:

$$f_n(x) = \begin{cases} 1 - nx & 0 \le x \le \frac{1}{n} \\ 0 & x \ge \frac{1}{n} \end{cases}$$



$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$



$\boxed{\text{POINTWISE LIMIT OF CTS FUNCTIONS NEED NOT BE CTS}}$

(a) $f_n \to f$ P.W, AS

LET $\varepsilon > 0$, $x \in (0, \frac{1}{n})$

Then $|f_n(x) - f(x)| = 1 - nx < \varepsilon$

provided $n > \frac{1-\varepsilon}{x} = N(\varepsilon, x)$.

(b) $f_n \not\to f$ UNIFORMLY as

$\|f_n - f\| = \sup\{1 - nx \mid x \in (0, \frac{1}{n})\} = 1 \not\to 0$.

THM 5

Suppose $f_n \to f$ UNIFORMLY and f_n is CTS then

The f is CTS.

PF

Fix $x \in X$. Let $\varepsilon > 0$.

Since $f_n \to f$ UNIF $\exists N = N(\varepsilon) : \forall y \in X$

$$|f_N(x) - f(x)| < \varepsilon/3$$

and $|f_N(y) - f(y)| < \varepsilon/3$

Also f_N is CTS So $\exists \delta > 0 : |x - y| < \delta$

implies

$$|f_N(x) - f_N(y)| < \varepsilon/3$$

So

$$|f(x) - f(y)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$
$$\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$
$$= \varepsilon$$

So f is CTS at x.

NOTICE In CEX 4 we almost have $f_n \to f$ UNIFORMLY

The only problem is NEAR $x = 0$.

In fact if we let $\varepsilon > 0$ and $E = [\varepsilon, 1]$

then ✓ $f_n \to f$ UNIFORMLY on E.

and $\lambda([0,1] \setminus E) = \lambda([0, \varepsilon)) = \varepsilon$ is small.

If $f_n \to f$ PW with f_n M'ble and $\mu(X) < \infty$ This always holds

EGOROV'S THM 6

Suppose $X \subseteq \mathbb{R}^N$ is measurable with $\lambda(X) < \infty$.

Let $f_n : X \to \mathbb{R}$ be a sequence of measurable f^{ns}

so that $f_n \to f$ PW on X

Then $\forall \varepsilon > 0$ $\exists$ m'ble set $B \subset X$ so that

(a) $\lambda(B) < \varepsilon$

(b) $f_n \to f$ UNIFORMLY on $X \sim B$.

PROOF

Fix k, n.

LET

$$B_k(n) = \bigcup_{m=k}^{\infty} \{x \in X \,/\, |f_m(x) - f(x)| \geq \tfrac{1}{n}\}$$ BAD SETS

CLAIM I $B_k(n)$ IS MEASURABLE

Pf Since $f_n \xrightarrow{PW} f$ we know f is M'ble by MF6

So $|f_m - f|$ is m'ble

So $B_k(n)$ is a countable union of m'ble sets and hence is m'ble. D

CLAIM II

ⓐ $B_k(n) \supseteq B_{k+1}(n) \cdots\cdots$ DECREASING

ⓑ $\bigcap_{k=1}^{\infty} B_k(n) = \emptyset$

PF ⓐ ✓

ⓑ Suppose $x \in \bigcap_{k=1}^{\infty} B_k(n)$

Then $x \in B_k(n)$ $\forall k$.

So $\forall m \geq k$

$$|f_m(x) - f(x)| \geq \frac{1}{n}.$$

So $f_m(x) \not\to f(x)$. $\times$

CLAIM III

$$\lim_{k\to\infty} \lambda(B_k(n)) = 0$$

PF From (M6) since $\lambda(B_1(n)) \leq \lambda(X) < \infty$.
we know by Claim II that

$$\lim_{k\to\infty} \lambda(B_k(n)) = \lambda\left(\bigcap_{k=1}^{\infty} B_k(n)\right) = \lambda(\emptyset) = 0.$$

By Claim III $\forall n \; \exists k_n$:

$$\lambda\left(B_{k_n}(n)\right) < \frac{\varepsilon}{2^n}$$

LET $\boxed{B = \bigcup_{n=1}^{\infty} B_{k}(n)}$ M'ble ✓

By Countable Additivity

$$\lambda(B) < \varepsilon$$ ✓

<u>SUPPOSE $x \notin B$</u>

Then $\forall n, \; x \notin B_{k_n}(n)$

$\therefore$ $\forall n \; \exists k_n : \; \forall m > k_n, \quad \forall x \notin B$

$$|f_m(x) - f(x)| < \frac{1}{n}$$

THIS just says

$$f_n \to f \text{ UNIF on } X \sim B.$$ ✓

□

III POINTWISE ALMOST EVERYWHERE (PW a.e) CONVERGENCE

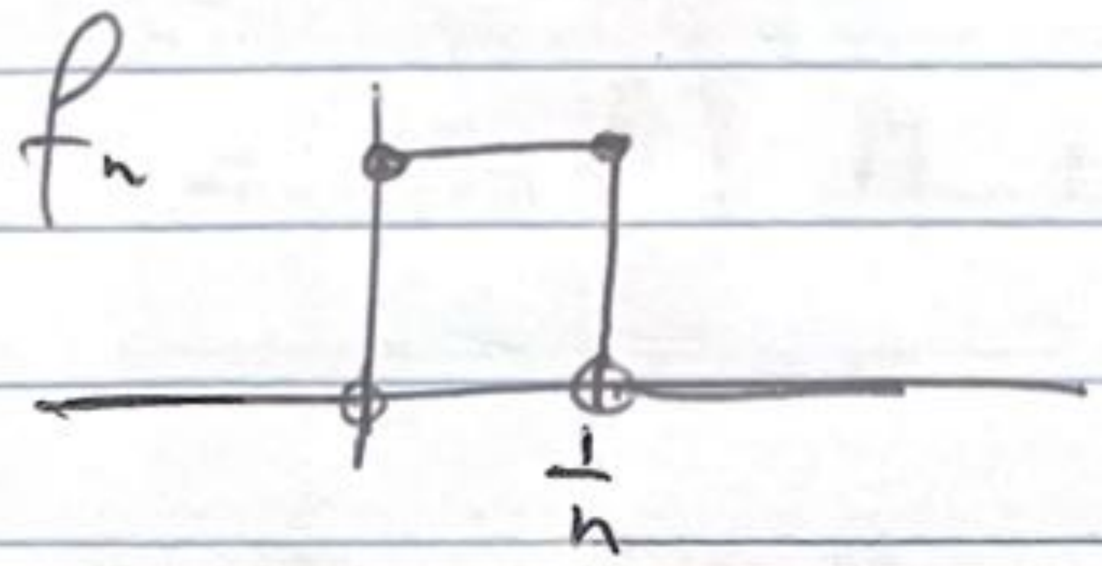
DEF 7 Let $f_n, f : X \to [-\infty, \infty]$ be measurable. We say $f_n \to f$ PW a.e if $\exists$ M'ble $Z \subset X$:

(a) $\lambda(Z) = 0$

(b) $\forall x \in X \sim Z$ $\quad f_n(x) \to f(x)$ as $n \to \infty$.

NOTATION " $f_n \to f$ λ a.e. "

EX 8 LET $f_n(x) = \chi_{[0, \frac{1}{n}]}(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$



Then

$f_n \to 0$ a.e.

but $f_n \not\to 0$ as $f_n(0) = 1 \ \forall n$.

EGOROV'S THM VERSION II (THM 9)

X Lebesgue m'ble.

$\lambda(X) < \infty$, $f_n : X \to \mathbb{R}$ Lebesgue measurable,

$f_n \to f$ PW a.e. where f is finite.

Then $\forall \varepsilon > 0$ $\exists$ closed $F \subset X$:

~~ⓐ $\lambda(F) < \infty$~~

ⓐ $\lambda(X \sim F) < \varepsilon$

ⓑ $f_n \to f$ UNIFORMLY on F.

(IV) L^∞-CONVERGENCE (UNIFORM a.e.)

DEF 10 Let $X \subseteq \mathbb{R}^N$, $f : X \to [-\infty, \infty]$ measurable.

Define

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in X} |f(x)|$$

$$= \inf \{ M : |f(x)| \le M \ \ \lambda\text{-a.e.} \}$$

EX ~~#1~~

ⓐ $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

$\|f\|_\infty = 0$ even tho $\sup_{x \in \mathbb{R}} |f| = 1$.

ⓑ $f = \chi_{\text{CANTOR}}$, $\|f\|_\infty = 0$ as $\lambda(C) = 0$

PROP 12

Let $f: \mathbb{R}^N \to \mathbb{R}$ be CTS.

Then

$$\|f\|_\infty = \sup\{ |f(x)| \mid x \in \mathbb{R} \}$$

PF HWK.

PROP 13 If $f \in L^\infty(X)$ Then $|f(x)| \leq \|f\|_\infty$ a.e $x \in X$

NOTE This means $\exists$ smallest M: $|f(x)| \leq M$ a.e.
This M is called $\|f\|_\infty$.
So the inf in def^n is attained.

PF By def^n of inf $\exists M_k \to \|f\|_\infty$ so that

$|f(x)| \leq M_k$ a.e.

So $\exists$ null set N_k: $|f(x)| \leq M_k$ $\forall x \in X \setminus N_k$

Define $N = \bigcup_{k=1}^{\infty} N_k$. Clearly $\lambda(N) = 0$

And if $x \notin N$ Then $|f(x)| \leq M_k$ $\forall k$
So $|f(x)| \leq \|f\|_\infty$.

$\square$

DEF 14 Let $X \subseteq \mathbb{R}^n$ be measurable.
Define

$$L^\infty(X) = \{ f : X \to [-\infty, \infty] \,/\, f \text{ M'ble}, \ \|f\|_\infty < \infty \}$$

We say $f = g$ in $L^\infty(X)$ iff $f = g$ a.e.

NOTE The functions in L^∞ are called essentially bounded.

THM 15 $L^\infty(X)$ is a complete normed vector space (ie a BANACH Space)

In particular

ⓐ $\|cf\|_\infty = |c| \, \|f\|_\infty$

ⓑ $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

ⓒ $\|f\|_\infty = 0 \iff f = 0$ in $L^\infty(X)$
(ie $f = 0$ a.e)

ⓓ Every Cauchy sequence in $L^\infty(X)$ converges ~~to~~ an elt of $L^\infty(X)$

PROP 16 IF $\|f_k - f\|_\infty \to 0$ as $k \to \infty$
THEN $f_k \to f$ P.W. a.e.

PF HWK

PROOF OF COMPLETENESS

Suppose f_k is Cauchy in $L^\infty(X)$.

So (letting $\varepsilon = \frac{1}{n}$)

$\forall m \ \exists N(m) : \quad \forall j, k \geq N(m)$

$$\| f_j - f_k \|_\infty \leq \frac{1}{m}$$

So $\exists$ Null set $N_{j,k,m}$:

$$| f_j(x) - f_k(x) | \leq \frac{1}{m} \qquad \forall x \in N_{j,k,m}^c . \quad (*)$$

~~Since~~
Let $\quad N = \bigcup_{m=1}^{\infty} \bigcup_{j,k \geq N(m)} N_{j,k,m}$ NULL SET ✓✓

Let $x \in N^c$.

Then by completeness of $\mathbb{R}$, $\exists f(x)$:

$$f_k(x) \longrightarrow f(x).$$

Since f_k are M'ble, f is M'ble.

By $(*)$ $\forall x \in N^c \qquad \forall j > N(m)$

$$| f_k(x) - f(x) | \leq \frac{1}{m} \qquad (**)$$

So $f_k - f \in L^\infty$

Since $f_k \in L^\infty$ and L^∞ is a vector space

$$f = (f - f_k) + f_k \in L^\infty.$$

Finally (**) says $f_k \to f$ in L^∞ $\square$

<u>EX 17</u>

Let $Z_1 \subseteq Z_2 \subseteq \ldots$ be a nested sequence of measure zero sets.

<u>EG</u> Let $\{r_k\}_{k=1}^\infty$ enumerate $\mathbb{Q}$ and set

$$Z_n = \{r_1, \ldots, r_n\}$$

Define

$$f_n(x) = \begin{cases} \frac{1}{n} & x \notin Z_n \\ n & x \in Z_n. \end{cases}$$

Now $\|f_n\|_\infty = \frac{1}{n} \to 0$ as $n \to \infty$

So $f_n \to 0$ in $L^\infty(\mathbb{R})$

<u>But</u> If $x \in Z = \bigcup_{n=1}^\infty Z_n$ Then $f_n(x) = n \to \infty$ as $n \to \infty$, So $f_n \not\to 0$ PW.