LECTURE 16 σ-ALGEBRAS AND MEASURABLE FUNCTIONS

DEFNS 1 Let X be a set (Think $X = \mathbb{R}^N$)

1. $\mathcal{P}(X)$ = Collection of all subsets of X including $\emptyset$ and X

2. An ALGEBRA of subsets of X is a collection of sets $\mathcal{M} \subset \mathcal{P}(X)$ so that

   a. $\emptyset \in \mathcal{M}$
   b. $A, B \in \mathcal{M} \implies A \cup B \in \mathcal{M}$
   c. $A \in \mathcal{M} \implies A^c = X - A \in \mathcal{M}$.

3. An algebra $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-ALGEBRA if

$$\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{M} \implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}.$$

NOTE By de Morgans Laws can replace $\cup$ by $\cap$ in 2, 3.

EXS
1. $\mathcal{M} = \mathcal{P}(X)$
2. $\mathcal{M} = \mathcal{L}$ = Lebesgue M'ble subsets of $\mathbb{R}^n$

LEMMA 2

Suppose $\{\mathcal{M}_i\}_{i\in I}$ is a collection of σ-algebras in X.

Then so is
$$\mathcal{M} = \bigcap_{i\in I} \mathcal{M}_i$$

~~This means~~ Let $A \subseteq X$ Then

$$A \in \mathcal{M} \iff A \in \mathcal{M}_i \quad \forall i \in I.$$

~~A CONSTRUCTION~~

DEF 3 Let $\mathcal{N} \subset P(X)$ be arbitrary.

The <u>σ-algebra generated by $\mathcal{N}$</u> is the intersection of all σ-algebras containing $\mathcal{N}$.

- It is also a σ-algebra by Lemma 2
- It is the smallest σ-algebra containing $\mathcal{N}$.

DEF 4 The class of <u>BOREL SETS</u>, $\mathcal{B}$, is the σ-algebra generated by the open sets in $\mathbb{R}^n$.

LEMMAS ① $\mathcal{B} \subset \mathcal{L}$

② All open sets are Borel, ③ All closed sets are Borel

PF

① $\mathcal{L}$ is a σ-algebra cont^g open sets

So by Def^n 3, $\mathcal{B} \subset \mathcal{L}$. □

DEF 6 If $A \subset \mathbb{R}^n$ is measurable and $\lambda(A) = 0$ we say A is a NULL set.

NOTE A null $\iff \lambda^*(A) = 0$

as if $\lambda^*(A) = 0$ Then

$$0 \le \lambda_*(A) \le \lambda^*(A) = 0$$

gives $\lambda_*(A) = \lambda^*(A)$

So A is measurable and $\lambda(A) = 0$.

THM 7 Let A be measurable

Then $\exists$ Disjoint Borel set E and null N so that

$$A = E \cup N.$$

PF By (TH 9) $\forall k \in \mathbb{N}$ $\exists$ Closed $F_k \subset A$ with

$$\lambda(A \sim F_k) < \frac{1}{k}$$

As $A \sim F_k \subset G_k \sim F_k$

Let $E = \bigcup_{k=1}^{\infty} F_k \in \mathcal{B}$.

Then $E \subset A$ and $N = A \sim E$ has

$$\lambda(N) = \lambda(A \sim E) < \lambda(A \sim F_k) < \frac{1}{k} \quad \forall k$$

as $F_k \subset E \Rightarrow A \sim E \subset A \sim F_k$.

So $\lambda(N) = 0$.

<u>THM 8</u> LET $f: \mathbb{R}^n \to \mathbb{R}^m$ be CTS.

If $A \subset \mathbb{R}^m$ (target space) is Borel

Then $f^{-1}(A) \subset \mathbb{R}^n$ is Borel.

<u>NOTE</u> Similar to $f^{-1}(\text{Open}) = \text{Open}$ if f CTS.

<u>PF</u> [OMIT]

Let $\mathcal{B}_m$ = Borel sets in $\mathbb{R}^m$

Define

$$\mathcal{M} = \{ A \subset \mathbb{R}^m \mid f^{-1}(A) \in \mathcal{B}_n \}$$

<u>CLAIM</u> $\mathcal{B}_m \subset \mathcal{M}$

Given claim: If $A \in \mathcal{B}_m$ then $A \in \mathcal{M}$
so $f^{-1}(A) \in \mathcal{B}_n$ ✓

PF OF CLAIM

By defn

$\mathcal{B}_n$ is smallest σ-algebra cont^g open subsets of $\mathbb{R}^m$.

So NTS

(A) $\mathcal{M}$ is a σ-algebra

(B) $\mathcal{M}$ contains all open subsets of $\mathbb{R}^m$.

PFS

(A) (1) $\emptyset \in \mathcal{M}$ as $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}_n$

(2) If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$ Then $f^{-1}(A_k) \in \mathcal{B}_n \ \forall k$

So $\displaystyle\bigcup_{k=1}^{\infty} f^{-1}(A_k) \in \mathcal{B}_n$

But Then $\displaystyle f^{-1}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \bigcup_{k=1}^{\infty} f^{-1}(A_k) \in \mathcal{B}_n$

So

$\displaystyle\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}.$

(3) Let $A \in \mathcal{M}$. Then $f^{-1}(A) \in \mathcal{B}_n$.

Now

$f^{-1}(A^c) = \mathbb{R}^n \sim \underbrace{f^{-1}(A)}_{\in \mathcal{B}_n}$ (V☐✓)

So $A^c \in \mathcal{M}$ too

(B) Let $G \subset \mathbb{R}^n$ be open
Then $f^{-1}(G)$ is open in $\mathbb{R}^n$ as f is CTS,

So $f^{-1}(G) \in \mathcal{B}_n$
So $G \in \mathcal{M}$.

So $\mathcal{M}$ contains all open sets in $\mathbb{R}^m$.

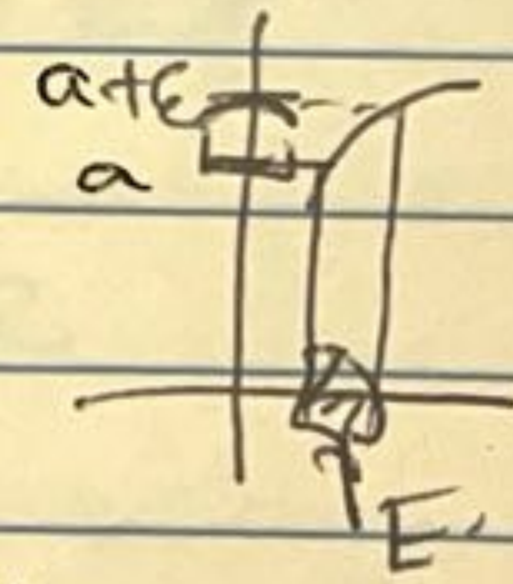
THM 9 $\mathcal{B} \subsetneq \mathcal{L}$.

PF See Jones P 110, USES $\exists$ Nonmeasurable Set and a version of Devils Staircase function.

MEASURABLE FUNCTIONS

LET $f: \mathbb{R}^n \to [-\infty, +\infty]$
Later we will define Lebesgue Integral in terms of sets of form

$$E = \{x \in \mathbb{R}^n / a \le f(x) < a + \varepsilon\}$$



$$E = f^{-1}([a, a+\varepsilon)) \subset \mathbb{R}^n$$

We would like E to be Lebesgue Measurable.

Since not always true we make the

DEF 10

Let $f: \mathbb{R}^n \to [-\infty, +\infty]$ and let m be a σ-algebra for $\mathbb{R}^n$.

We say f is m_Measurable if

$\forall t \in [-\infty, \infty]$ we have $f^{-1}([-\infty, t]) \in m$.

NOTE
$m = \mathcal{B}$: f is Borel M'ble
$m = \mathcal{L}$: f is Lebesgue M'ble.
$\cdot\, \mathcal{B} \subset \mathcal{L} \Rightarrow$ EVERY BOREL M'BLE FUNCTION IS LEBESGUE MEASURABLE

PROP 11

① If f is CTS Then f is Borel M'ble.

② Let $\chi_A(x) = \begin{cases} 0 & \text{IF } x \notin A \\ 1 & \text{IF } x \in A \end{cases}$

be CHARACTERISTIC FN of set $A \subset \mathbb{R}^n$

Then $\chi_A : \mathbb{R}^n \to \mathbb{R}$ is m_M'ble

iff $A \in m$.

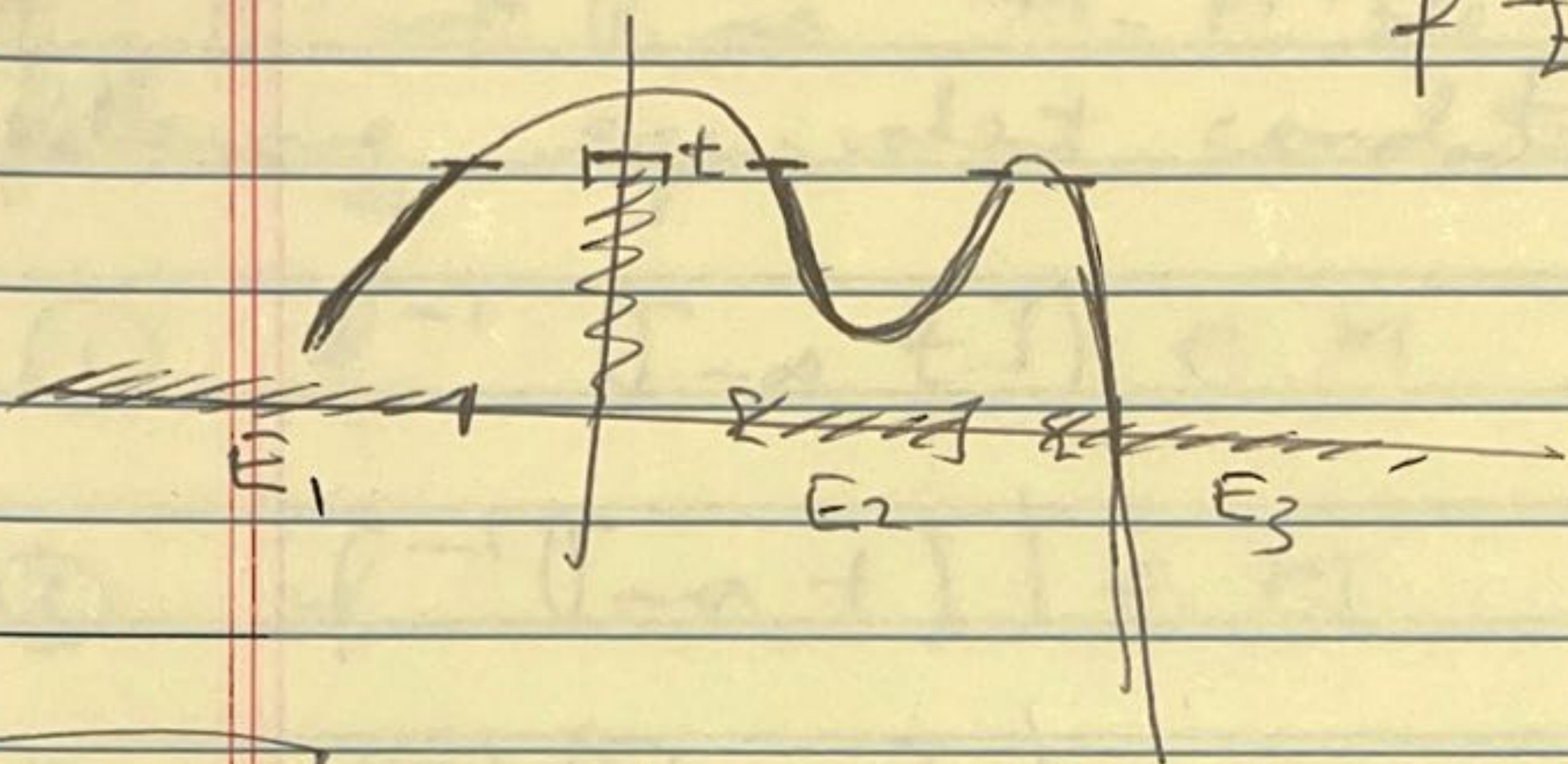
~~SKIP TO THM pII~~

SINCE $\mathcal{B} \subsetneq \mathcal{L}$

COR $\exists$ Lebesgue M'ble f that is not Borel M'ble

PICTURE $f : \mathbb{R} \to \mathbb{R}$

$$f^{-1}[-\infty, t] = E_1 \cup E_2 \cup E_3$$



LEMMA 12

~~RECALL~~ Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

(0) $f^{-1}(A) = \{ x \in \mathbb{R}^n \,/\, f(x) \in A \}$

(1) $f^{-1}\left(\bigcup_{k \in K} A_k \right) = \bigcup_{k \in K} f^{-1}(A_k)$

(2) $f^{-1}\left(\bigcap_{k \in K} A_k \right) = \bigcap_{k \in K} f^{-1}(A_k)$

So

$$\mathbb{R}^n = f^{-1}(A \cup A^c) = f^{-1}(A) \cup f^{-1}(A^c)$$

$$\emptyset = f^{-1}(A \cap A^c) = f^{-1}(A) \cap f^{-1}(A^c)$$

So

(3) $\boxed{f^{-1}(A^c) = \mathbb{R}^n \sim f^{-1}(A)}$

TECHNICAL LEMMA 13

$f : \mathbb{R}^n \to \mathbb{R}$ is m-M'ble iff one of following equivalent conditions holds

① $f^{-1}([-\infty, t]) \in m \quad \forall t \in [-\infty, \infty]$

② $f^{-1}([-\infty, t)) \in m \quad \forall t \in (-\infty, \infty]$

③ $f^{-1}([t, \infty]) \in m \quad \forall t \in [-\infty, \infty]$

④ $f^{-1}((t, \infty]) \in m \quad \forall t \in [-\infty, \infty)$

⑤ $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ belong to m

and $f^{-1}(E) \in m \quad \forall$ Borel $E \subset \mathbb{R}$.

PFS

① ⇒ ④ : $f^{-1}((t, \infty]) = \mathbb{R}^n \sim f^{-1}([-\infty, t])$ by L12 ③
$\in m$ ✓

① ⇒ ② : $[-\infty, t) = \bigcup_{r < t} [-\infty, r]$

So

$$f^{-1}[-\infty, t) = \bigcup_{r<t} f^{-1}[-\infty, r] \in m$$ by L12 ① and m a σ-alg.

5 $\Rightarrow$ 1 $\quad E = [-\infty, t]$ is Borel.

1 $\Rightarrow$ 5

Let $S = \{ E \subset \mathbb{R} \,/\, f^{-1}(E) \in m \}$

a $\;S$ is a σ-algebra. (as before)

b $\;S$ contains all open subsets of $\mathbb{R}$.

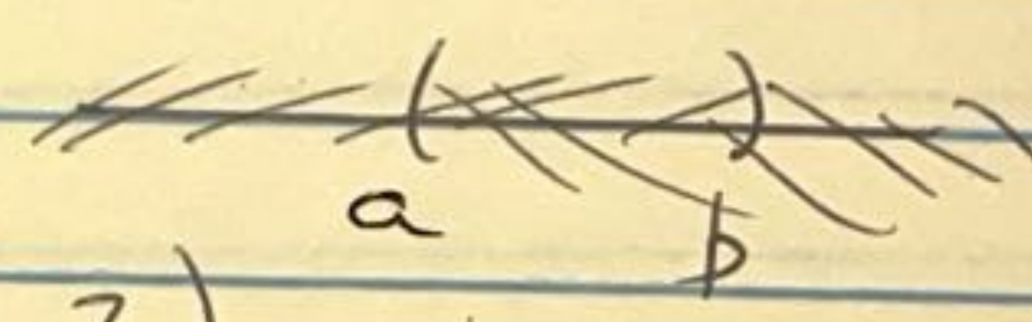
Given a, b we conclude $\mathcal{B} \subset S$

i.e. $\forall$ Borel $E \subset \mathbb{R}$, $f^{-1}(E) \in m$ as req'd.

PROOF OF b

FACT Every open subset of $\mathbb{R}$ is a countable union of open intervals

So if $G \subset \mathbb{R}$ is open then $G = \bigcup_{k=1}^{\infty} (a_k, b_k)$



$$f^{-1}(G) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, b_k)$$

$$= \bigcup_{k=1}^{\infty} f^{-1}\left([-\infty, b_k) \cap (a_k, \infty]\right)$$

$$= \bigcup_{k=1}^{\infty} f^{-1}[-\infty, b_k) \cap f^{-1}((a_k, \infty]) \in m$$

by 3, 4 as m is a σ-algebra $\square$

THM 14

Let $f, g : \mathbb{R}^n \to \cancel{[-\infty,\infty]}\ \mathbb{R}^1$ be m–M'ble

Then

MF1 If $\phi : \mathbb{R} \to \mathbb{R}$ is Borel M'ble
Then $\phi \circ f : \mathbb{R}^n \to \mathbb{R}$ is m–M'ble.

MF2 If $f \neq 0$ Then $\frac{1}{f}$ is m–M'ble

MF3 Let $0 < p < \infty$. Then $|f|^p$ is m'ble

MF4 $f+g$ is m'ble

MF5 fg ______

MF6 If $f_k : \mathbb{R}^n \to [-\infty, \infty]$ are m–M'ble
Then so are

$\sup f_k$, $\inf f_k$

$\limsup f_k$ $\liminf f_k$

$\lim f_k$ if $\exists$

PF

1. Use L13 5 applied to $\phi \circ f$

SB $\forall$ Borel $E \subset \mathbb{R}$ we have $(\phi \circ f)^{-1}(E) \in \mathcal{M}$.

Well

$$(\phi \circ f)^{-1}(E) = f^{-1}(\phi^{-1}(E))$$

and since ϕ is Borel M'ble, $\phi^{-1}(E) \in \mathcal{B}$.

So by L13 5 again, This time applied to f we have

$$(\phi \circ f)^{-1}(E) = f^{-1}(\phi^{-1}(E)) \in \mathcal{M}. \checkmark$$

3. $\phi(t) = |t|^p$ is CTS and hence Borel M'ble

$$|f|^p = \phi \circ f. \quad \text{Apply } 1.$$

4. We must show $\forall t \in \mathbb{R}$ That

$$\{x \mid f(x) + g(x) < t\} \in \mathcal{M} \text{ ~~is M M'ble~~}.$$

Well $\quad f(x) + g(x) < t \iff f(x) < t - g(x)$

$\iff \exists r \in \mathbb{Q} :$

$f(x) < r < t - g(x)$.

$\iff \exists r \in \mathbb{Q}:$

$f(x) < r$ and $g(x) < t - r$.

$$\overset{\exists r \in \mathbb{Q}}{\Longleftrightarrow} \quad x \in f^{-1}(-\infty, r) \cap \cancel{\ } g^{-1}(-\infty, t-r)$$

$$\Longleftrightarrow \quad x \in \bigcup_{r \in \mathbb{Q}} f^{-1}(-\infty, r) \cap g^{-1}(-\infty, t-r)$$

So

$$\{x \mid f(x)+g(x) < t\} = \bigcup_{r \in \mathbb{Q}} f^{-1}(-\infty, r) \cap g^{-1}(-\infty, t-r)$$

$\in \mathcal{M}$ as f, g M'ble.

5 $fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$

Now use 3 + 4

6 Must show $\forall t \in \mathbb{R}$ that

$$\{x \mid \sup_k f_k(x) \le t\} \in \mathcal{M}$$

~~Well~~ Fix x

$$\cancel{\sup_{LUB} f_k(x) \le t \Longleftrightarrow \forall k \quad f_k(x) \le t}$$

CLAIM $\{x \mid \sup_k f_k(x) \le t\} = \bigcap_k \underset{\in \mathcal{M}}{\{x \mid f_k(x) \le t\}} \in \mathcal{M}$

PF Fix t

$\boxed{\subseteq}$ Suppose x : $\underbrace{\sup_k f_k(x)}_{LUB} \le \underbrace{t}_{UB}$

Then $f_k(x) \le t \quad \forall t$

So $x \in \bigcap_k \{x / f_k(x) \le t\}$

$\boxed{\supseteq}$ If $f_k(x) \le \underbrace{t}_{UB} \quad \forall k$

Then

$$\underbrace{\sup_k f_k(x)}_{LUB} \le t$$

D.

DEF 15 A SIMPLE FUNCTION $s : \mathbb{R}^n \to [-\infty, \infty]$ is a function that assumes only a finite # of values.

If the values are $\alpha_1 \ldots \alpha_m$ and

$$A_k = \{x \in \mathbb{R}^n / s(x) = \alpha_k\} = s^{-1}(\{\alpha_k\})$$

Then

$$\boxed{s = \sum_{k=1}^{m} \alpha_k \chi_{A_k}}$$

LEMMA 16

s is $\mathfrak{M}$ – M'ble $\iff$ $A_k \in \mathfrak{M} \quad \forall k \in \{1, \ldots, m\}$.

~~SKIP TO THM 19. OMIT PF~~ THINK!

DEF 17 ① Let $a \in [-\infty, \infty]$

Define

$$a_+ = \begin{cases} a & \text{IF } a \geq 0 \\ 0 & \text{IF } a < 0 \end{cases} \qquad a_+ \geq 0$$

$$a_- = \begin{cases} 0 & \text{IF } a \geq 0 \\ -a & \text{IF } a < 0 \end{cases} \qquad a_- \geq 0$$

Then

$$a = a_+ - a_-$$
$$|a| = a_+ + a_-$$
$$a_+ a_- = 0$$

② Let $f : \mathbb{R}^n \to [-\infty, \infty]$

Define

$$f_\pm : \mathbb{R}^n \to [-\infty, \infty] \text{ by}$$

$$f_\pm(x) = (f(x))_\pm$$

LEMMA 18 If f is m'ble so are $f_\pm$.

THM 19 Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be m M'ble

Then $\exists$ seq s_k of m M'ble simple f^{ns}:

$$f = \lim_{k \to \infty} s_k$$

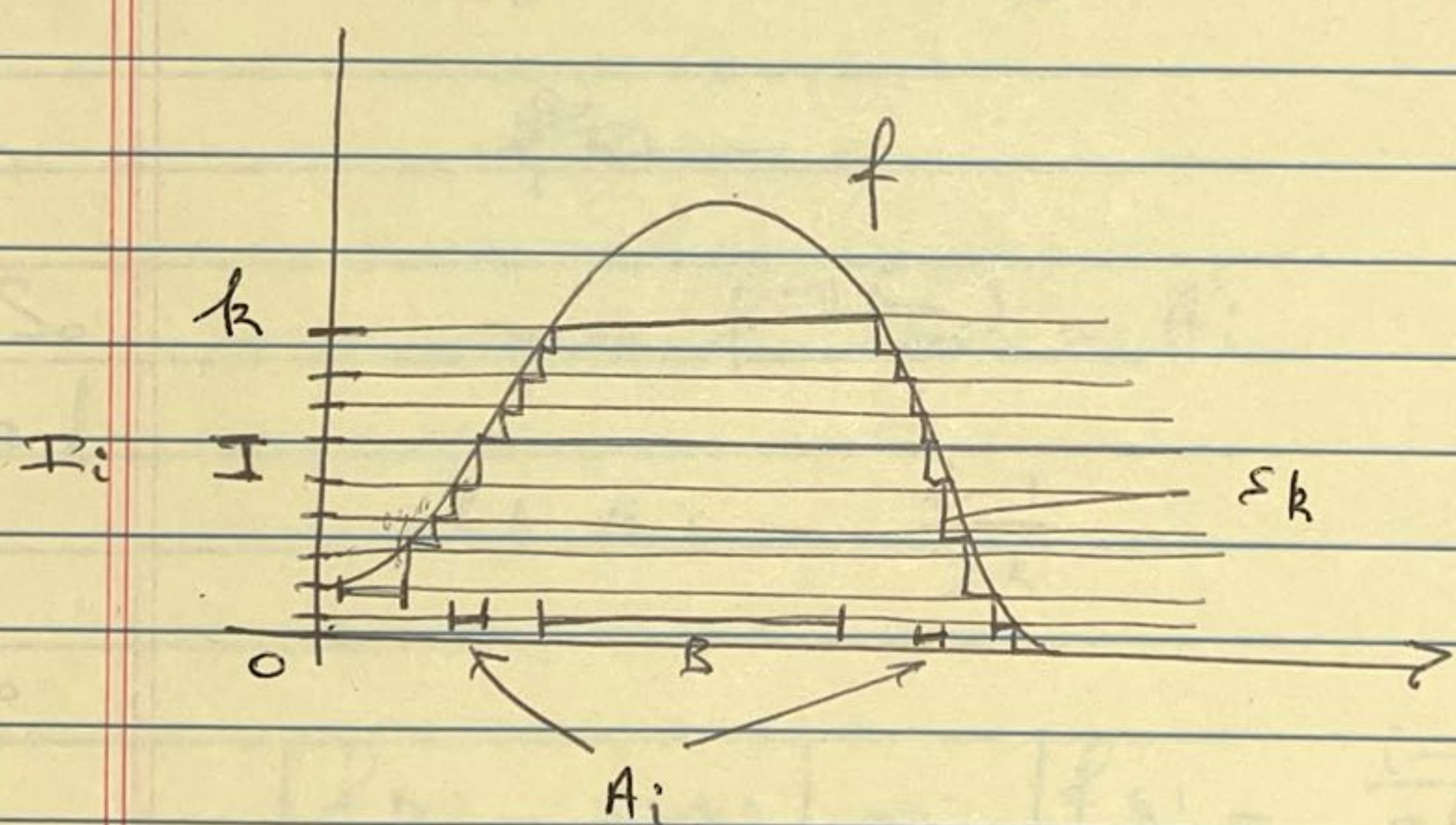
and $s_k \leq f \; \forall k$ and $s_1 \leq s_2 \leq \ldots$

PF $\boxed{\text{Case } f \geq 0}$

Define s_k as follows.

Divide $[0, k]$ into $k2^k$ subintervals of length $1/2^k$

s_0 $\quad I_i = \left[\frac{i-1}{2^k}, \frac{i}{2^k} \right)$ $\quad i = 1, 2, \ldots, k2^k$



Let $\quad A_i = f^{-1}(I_i) \in \mathcal{M}$

$\quad B = f^{-1}([k, \infty]) \in \mathcal{M}$

FORMULA

$$s_k = \sum_{i=1}^{k2^k} \frac{i-1}{2^k} \chi_{A_i} + k \chi_B.$$

To prove $s_k \to f$.

LET $\varepsilon > 0$. and $x \in \mathbb{R}^n$.

Pick N : $N > -\log_2 \varepsilon$ and $f(x) < N$

Then $\frac{1}{2^k} < \varepsilon$

and $\exists I_i$: $f(x) \in I_i$

~~So~~ ~~$f(x)$~~

So $x \in f^{-1}(I_i) = A_i$

and $s_k(x) = \frac{i-1}{2^k}$

AS $f(x) \in I_i$

$\therefore$ $|f(x) - s_k(x)| = |f(x) - \frac{i-1}{2^k}| < \frac{1}{2^k} < \varepsilon$

$\square$

$\boxed{\text{GENERAL CASE}}$

Write $f = f_+ - f_-$ $f_\pm \geq 0$

Apply Previous Case to $f_\pm$ and use fact sum of simple f^{ns} is simple.