[J, #1]

LECTURE 12 — THE TOPOLOGY OF $\mathbb{R}^n$

(SUMMARY OF MAJOR DEFNS + RESULTS)

A. SETS

$$\mathbb{R}^n = \{ x = (x_1, \dots x_n) / x_j \in \mathbb{R} \}$$

Let $A, B \subseteq \mathbb{R}^n$.

Define ① $A^c = \{ x \in \mathbb{R}^n / x \notin A \}$ COMPLEMENT OF A

② $A \sim B = A \cap B^c = \{ x \in \mathbb{R}^n / x \in A, x \notin B \}$

③ Let I be an arbitrary set used for indexing. (EX $I = \mathbb{N} = \{1, 2, 3, \dots\}$)
Suppose $\forall i \in I$ $\exists$ Set $A_i \subseteq \mathbb{R}^n$.
Define

$$\bigcup_{i \in I} A_i = \{ x \in \mathbb{R}^n / \exists i \in I : x \in A_i \}$$

$$\bigcap_{i \in I} A_i = \{ x \in \mathbb{R}^n / \forall i \in I, x \in A_i \}$$

B. METRIC SPACE TOPOLOGY

DEF 1 $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$ EUCLIDEAN NORM ON $\mathbb{R}^n$.

Δ INEQ $\|x + y\| \leq \|x\| + \|y\|$

<u>METRIC ON $\mathbb{R}^n$</u> $\quad d(x,y) = \|x-y\| \quad$ = DISTANCE FROM x to y

<u>PROPERTIES OF METRIC</u>

(1) $d(x,y) \geq 0$

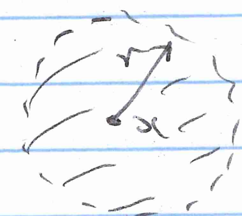
(2) $d(x,y) = 0 \iff x = y$

(3) $d(x,y) = d(y,x)$

(4) $d(x,y) \leq d(x,z) + d(z,x)$

<u>DEFS</u> (1) <u>OPEN BALL</u> center x, radius r is

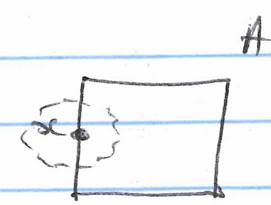
$$B(x,r) = \{ y \in \mathbb{R}^n / \ d(x,y) < r \}$$



(2) Let $x \in A \subseteq \mathbb{R}^n$. We say x is an <u>INTERIOR POINT</u> of A if $\exists r > 0$:

$$B(x,r) \subseteq A.$$

x IS INTERIOR PT



x NOT INTERIOR PT

3. Let $A \subseteq \mathbb{R}^n$. We say A is an <u>OPEN SET</u> if every pt of A is an interior point of A.

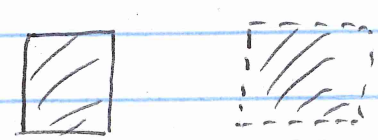
<u>PROP 3</u>

1. $\emptyset$ is OPEN
2. $\mathbb{R}^n$ is OPEN
3. Let $\{A_i\}_{i \in I}$ be an <u>arbitrary</u> collection of open sets. Then $\bigcup_{i \in I} A_i$ is OPEN
4. Let $\{A_i\}_{i=1}^{N}$ be a <u>finite</u> collection of open sets. Then $\bigcap_{i=1}^{N} A_i$ is OPEN.

<u>DEF 4</u> Let $A \subseteq \mathbb{R}^n$. The <u>INTERIOR</u> of A is

$$A^{\circ} = \{ x \in \mathbb{R}^n \, / \, x \text{ IS AN INTERIOR PT OF } A \}$$

<u>DEF 5</u> A subset A of $\mathbb{R}^n$ is <u>CLOSED</u> if A^c is open



A $\qquad$ A°

PROOF OF PROP 3

(1) Since $\emptyset$ does not have any points there is nothing to check!

(2) Let $x \in \mathbb{R}^n$. Then $B(x, 1) \subseteq \mathbb{R}^n$ ~~so~~ x is an interior point of $\mathbb{R}^n$. So all pts of $\mathbb{R}^n$ are interior points. So $\mathbb{R}^n$ is open by def^n.

(3) Let $B = \bigcup_{i \in I} A_i$, where each A_i is open.

Let $x \in B$. Then $\exists i : x \in A_i$

Since A_i is open $\exists r > 0 : B(x, r) \overset{*}{\subseteq} A_i$

CLAIM $B(x, r) \subseteq B$.

PF

Let $y \in B(x, r)$. Then $y \in A_i$ by $*$
So $y \in B$ by def^n of union.

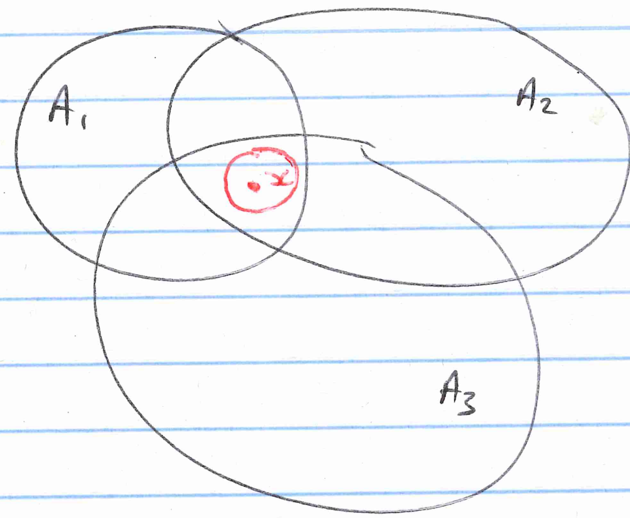
So $B(x, r) \subseteq B$. $\square$

So by claim Every $x \in B$ is an interior point of B

So B is open.

4 Let $B = \bigcap_{i=1}^{N} A_i$ where each A_i is open

PICTURE ($N = 3$)

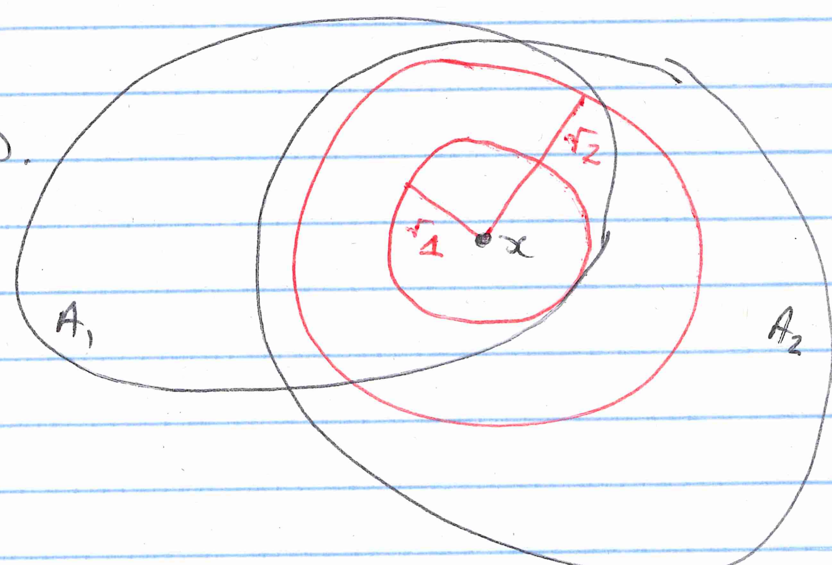


Let $x \in B$.
Then $x \in A_i$ for $i = 1 - N$.

Since A_i is open $\exists\, r_i > 0$: $B(x, r_i) \subseteq A_i$

Let
$r = \min\{r_1, \ldots, r_N\} > 0$.

as N is finite.



CLAIM
$B(x, r) \subseteq B$

Hence every pt of B is an interior pt and so B is open.

PF OF CLAIM

Since $r \le r_i \ \forall i$

$$B(x,r) \subseteq B(x, r_i) \subseteq A_i \qquad \forall i$$

So
$$B(x,r) \subseteq \bigcap_{i=1}^{N} A_i = B$$

[IF $y \in B(x,r)$ Then $y \in A_i \ \forall i$ So $y \in B$]

$\square$

PROP 6 Any open ball is an open set

PF HWK

PROP 7

A° = UNION OF ALL OPEN SUBSETS OF A

PF

$\boxed{\subseteq}$ Let $x \in A^\circ$. (x is an interior pt of A)
So by defn $\exists r > 0 : \ x \in B(x,r) \subseteq A$

Since $B(x,r)$ is open (PROP 6),
x belongs to an open subset of A and hence
to union of all open subsets of A.

$\boxed{\supseteq}$ Let $x \in$ Union of all open subsets of A

Then $\exists\, G \subseteq A$ open : $x \in G$.

Since G is open $\exists\, r > 0$: $B(x,r) \subseteq G \subseteq A$

So $x \in B(x,r) \subseteq A$.

So x is an interior pt of A, $x \in A^\circ$.

$\square$

C. COMPACT SETS

DEF 8 A set $A \subseteq \mathbb{R}^n$ is COMPACT if

whenever A is contained in a union of open sets

then A is contained in the union of finitely many of these sets

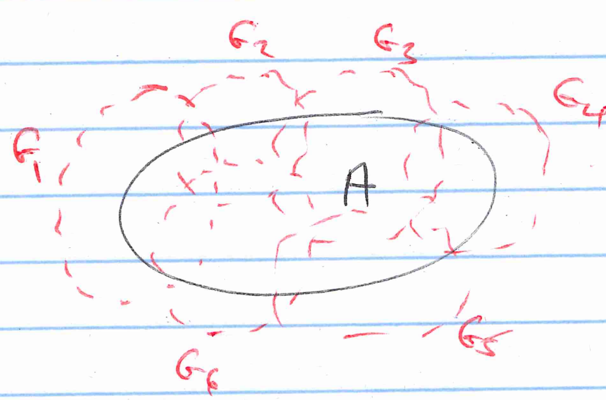
ie

If $A \subseteq \bigcup_{i \in I} G_i$ with each G_i open

Then $\exists\, i_1 \dots i_N \in I$: $A \subseteq \bigcup_{k=1}^{N} G_{i_k}$

TERMINOLOGY An OPEN COVER of A is $\{G_i\}_{i \in I}$ with

G_i open and $A \subseteq \bigcup_{i \in I} G_i$



So

A COMPACT means "Every Open Cover of A has a Finite Subcover"

PROP 9

1. $\emptyset$ is COMPACT
2. Any finite set is compact
3. If A, B are compact Then $A \cup B$ is compact
4. Any FINITE UNION of Compact Sets is Compact
5. $B(x, r)$ is NOT COMPACT
6. $\mathbb{R}^n$ is NOT COMPACT.

PF (See HWK FOR REST)

3. Let $C = A \cup B$.

Let $C \subseteq \bigcup_{i \in I} G_i$ where each G_i is open.

Now $A \subseteq C \subseteq \bigcup_{i \in I} G_i$ is an open cover for A

Since A is compact $\exists\, i_1 - i_K$ so That

$$A \subseteq \bigcup_{k=1}^{K} G_{i_k} \qquad (\exists \text{ Finite Subcover of } A)$$

SIMILARLY since B is compact $\exists\, i_{K+1} \ldots i_{K+L}$ so That

$$B \subseteq \bigcup_{k=K+1}^{K+L} G_{i_k}$$

So

$$C = A \cup B \subseteq \bigcup_{k=1}^{K+L} G_{i_k} \qquad (\exists \text{ Finite Subcover of } C)$$

So C is compact.

5) Let $G_n = B(x, r(1-\frac{1}{n}))$ open.

CLAIM

$$B(x,r) \subseteq \bigcup_{n=1}^{\infty} G_n.$$

PF

Let $y \in B(x,r)$.

Then $\|y - x\| = s < r$.

$\frac{s}{r} < 1$

$\Rightarrow \exists n : \frac{s}{r} < 1 - \frac{1}{n}$

Choose n so that $s < r(1-\frac{1}{n}) < r$

So

$$y \in B(x, r(1-\tfrac{1}{n})) = G_n.$$

$\square$

Now

Suppose $\exists$ finite subcover. Let N be largest n in subcover.

Since $G_{n-1} \subseteq G_n \quad \forall n$

We have

$$B(x,r) \subseteq G_N \quad \text{holding}$$

That implies

$$r < r(1-\tfrac{1}{N}) \quad \times$$

$\square$

So $B(x,r)$ cannot be compact.

HEINE – BOREL THM is

$$A \subseteq \mathbb{R}^n \text{ is COMPACT} \iff A \text{ is CLOSED + BOUNDED}$$

PF THAT COMPACT $\Rightarrow$ CLOSED

NOTE PF THAT COMPACT $\Rightarrow$ BOUNDED is similar to arguement below
PF THAT CLOSED + BOUNDED $\Rightarrow$ COMPACT is harder
and relies on completeness of $\mathbb{R}$.

Let A be compact.
NTS A^c is open.

Let $x \in A^c$.

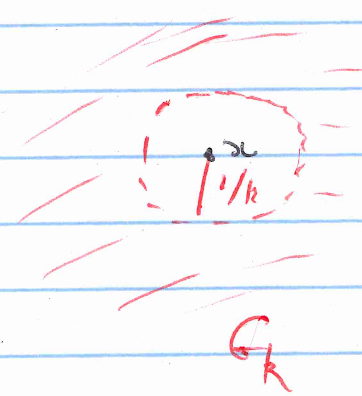
Define $G_k = \{ y \in \mathbb{R}^n \,/\, d(x,y) > \frac{1}{k} \}$ $\quad k = 1, 2, \ldots$

CLAIM (U∃✓)

① $G_k \subseteq G_{k+1}$ $\forall k$

② G_k OPEN

③ $\mathbb{R}^n \sim \{x\} = \bigcup_{k=1}^{\infty} G_k$



Hence

$$A \subseteq \mathbb{R}^n \sim \{x\} \subseteq \bigcup_{k=1}^{\infty} G_k$$

Since A is compact and $G_k \subseteq G_{k+1}$ $\forall k$ $\exists K$:

$$A \subseteq G_K$$

So $$G_K^c \subseteq A^c$$

ie $$\{y \mid d(x,y) \leq \tfrac{1}{k}\} \subseteq A^c$$

So $$x \in B(x, \tfrac{1}{k}) \subseteq A^c$$

So x is an interior pt of A^c.

So A^c is open.

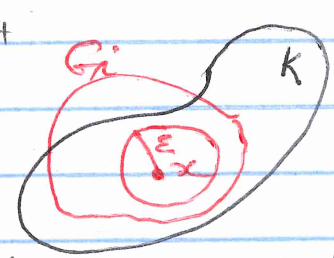
□

LEMMA 11

Let K be compact and $K \subseteq \bigcup_{i \in I} G_i$ with G_i open.

Then $\exists \varepsilon > 0$: $\forall x \in K$ $\exists i \in I$ WITH

$$B(x, \varepsilon) \subset G_i$$



ε = LEBESGUE # OF COVERING $\{G_i\}_{i \in I}$ OF K.

"THE G_i ARE NOT TOO THIN"

PF

A Let $x \in K$. $\exists\, i(x) : x \in G_{i(x)}$.

Since $G_{i(x)}$ is open $\exists\, r(x) > 0$:

$$B(x, 2r(x)) \subseteq G_{i(x)}. \qquad (1)$$

Since $$K \subseteq \bigcup_{x \in K} B(x, r(x))$$

and K is compact $\exists\, x_1 \dots x_N \in K$:

$$K \subseteq \bigcup_{k=1}^{N} B(x_k, r(x_k)) \qquad (2)$$

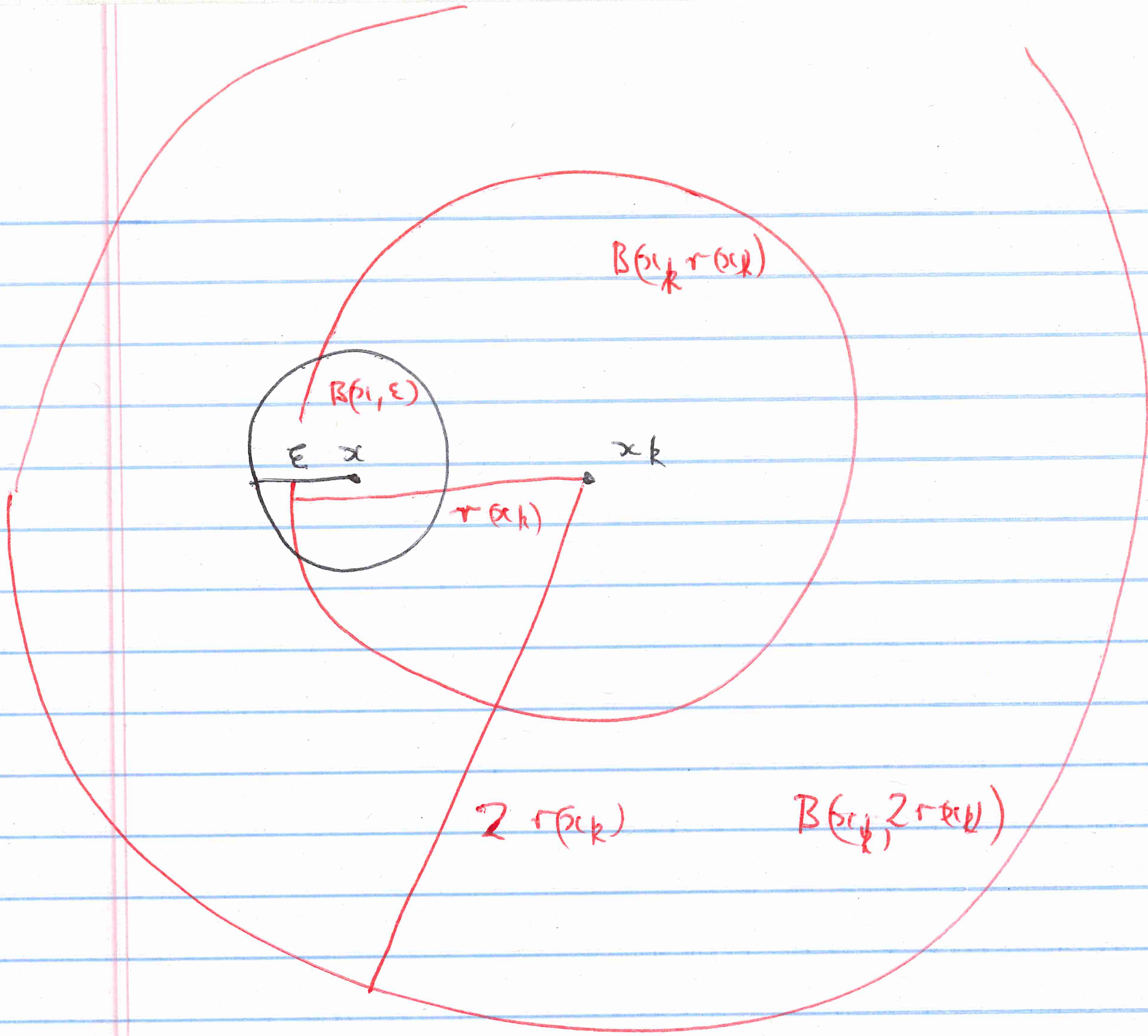
LET $$\varepsilon = \min\{r(x_k) \mid k = 1 \dots N\} > 0$$

B Let $x \in K$
Then by (2) $\exists\, k : x \in B(x_k, r(x_k))$.

So $d(x, x_k) < r(x_k) \leq 2r(x_k) - \varepsilon$ by def^n ε

Hence

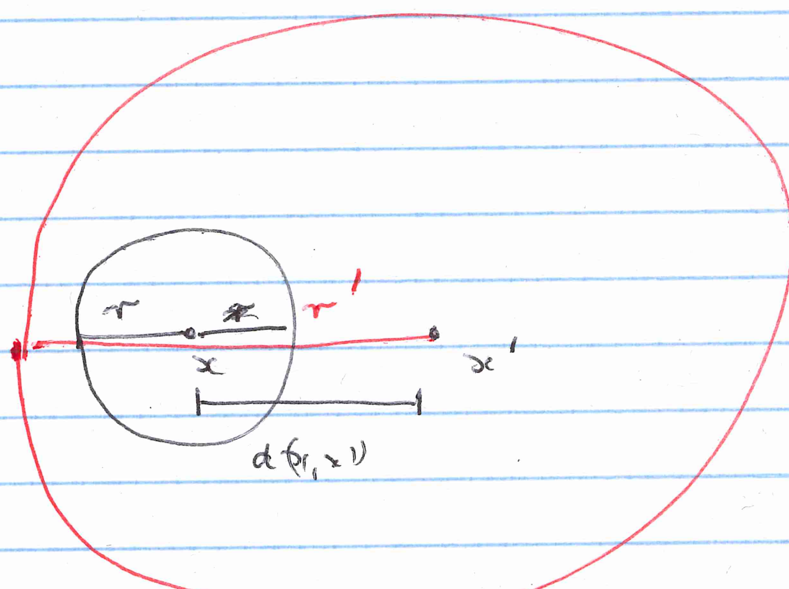
$$B(x, \varepsilon) \overset{*}{\subseteq} B(x_k, 2r(x_k)) \subseteq G_{i(x_k)}$$ ✓



REASON FOR ⊛

CLAIM $B(x,r) \subseteq B(x',r') \iff d(x,x') \leq r' - r$

HWK



$\boxed{d(x,x') + r \leq r'}$