Notes on Historical Context for Lebesgue Measure and Integration With Scaffolded Problems on Pathological Sets and Functions

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These notes are based on the fascinating text of Bressoud [1] and the chapter by Hochkirchen [2] in a fabulous book on the history of analysis edited by Jahnke. Both of these works owe a large debt to Hawkins seminar history of the development of Lebesgue's theory [3].

Bressoud starts out by quoting Luzin, one of the contributors to Lebesgue's theory. He argues that the rigor introduced into analysis by Weierstrass and Cauchy was necessary, but that there is a *stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work ... of their proofs.* He warns us never to get too comfortable with analysis that we do not *forget this stark contradiction*.

1 Integration Prior to Riemann

- In order to come up with a notion of $\int_a^b f(x)\,dx$ mathematicians needed to agree on what they meant by the concept of a function. This concept developed gradually from the time of Newton and Leibniz all the way into the latter half of the 19th century. The three stages in this development were (i) functions as algebraic formulae (ii) the concept of a continuous function independent of a formula but still related to graph of a function, and (iii) a rule that assigns an element of the target space to each element of the domain, which includes many pathological examples which are nearly impossible to visualize.

- In the 18th century the term integration referred to the problem of finding a formula for the anti-derivative of a function given by an algebraic formula. There was a tacit understanding that the formula $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative for f, enabled one to calculate area under a graph, distance from speed, work from force, and so on. However, mathematicians were very uncomfortable with the notion of an infinitesimal like dx and the concept of a limit. For example, the classic 1802 textbook of Lacroix provides no explicit definition of the integral, instead stating that "Integral calculus is the inverse of differential calculus. Its goal is to restore the functions from their differential coefficients.".

- It was only with the introduction of Fourier series in the early 1800's that mathematicians began to grapple with arbitrary functions that were not given by an algebraic formula. Questions about Fourier series spurred the development integration theories throughout the 1800s. To start with, Fourier defined the Fourier series coefficients of a "general" function, f, by

  $$a_n = \int_0^{2\pi} f(x)\cos(nx)\,dx \qquad \text{and} \qquad b_n = \int_0^{2\pi} f(x)\sin(nx)\,dx$$

  and the Fourier series of f by

  $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

  A big question became: What functions have Fourier series, and if aFourier series exists does it converge to f?

- In 1821 Cauchy introduced the ϵ–δ definition of a continuous function. This concept enabled people to think of functions as being simply given by their graphs. No formulae were required to analyze them. In 1823 he showed that if f is continuous, what we now call the left Riemann sum of f converges to the area under the graph of f as the partition size $\Delta x \to 0$. In other words, he showed there is a well defined notion of $\int_a^b f(x)\,dx$ whenever f is continuous. Hence any continuous function has a Fourier series.

- Dirichlet (1829) showed that if f is monotone with at most a finite number of discontinuities at which the value of f at each discontinuity equals the average of the left and right limits of f then f can be integrated (in the sense of Cauchy) and hence f has a Fourier series that converges pointwise to f. In their quest to better understand which functions can be integrated this result led researchers such as Dirichlet's student Riemann to focus on functions that oscillate.

- Dirichlet (1829) further generalized the notion of function to be a rule $f : X \to Y$ that assigns a value $f(x) \in Y$ for every $x \in X$ (although he probably didn't say it quite that way). Still, he used this concept to define the famous Dirichlet function $f : [0, 1] \to [0, 1]$ that takes the value 1 on the rationals and zero on the irrationals. This function oscillates a lot! It is super wierd. You can't draw a picture of it. Can you make sense of $\int_a^b f(x)\,dx$ for it? It turns out that Riemann says no but Lebesgue says yes! However the FTC does not hold for this function.

- Bressoud discusses the two version of the FTC as follows.

  Theorem 1.1 (FTC I)**.** *If f is the derivative of F then under suitable hypotheses*

  $$\int_a^b f(t)\,dt \;=\; F(b) - F(a). \tag{1}$$

  and

  Theorem 1.2 (FTC II)**.** *If f is integrable then under suitable hypotheses*

  $$\frac{d}{dx}\int_a^x f(t)\,dt \;=\; f(x). \tag{2}$$

  The earliest reference Bressoud knows of to FTC I is due to Poisson (1820), whose proof is wanting in several respects. Cauchy (1823) gives a rigorous proof of FTC II based on his definition of the definite integral. However, the result is not prominently highlighted in the form of a theorem.

2 The Riemann Integral

- Riemann developed his theory of integration in a 5-6 page chapter of his PhD thesis on Fourier series. He goes beyond Cauchy, who only considered continuous functions, by introducing the new concept of a Riemann integrable function. In particular he works with arbitrary tagged partitions, rather than simply left-sums since that simplifies the proofs. Riemann understood that integrable functions must be bounded, but he allowed improper integrals as a work around.

- In addition to the very brief lecture on the Riemann integral in our MATH 6301 course, for technical details on Riemann integrability I recommend Lectures 1-6 of the MATH 5302 course I taught in the past.

- Riemann introduces the concept of a tagged partition, (P, ξ), where the tags, ξ_i are points in the subintervals, $[x_{i-1}, x_i]$ of P at which we will evaluate the function to get the height of a rectangle. This yields the Riemann sum

$$\sigma(f, P, \xi) \;=\; \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i=1}). \tag{3}$$

Riemann says a bounded function : $[a, b] \to \mathbb{R}$ is integrable if the Riemann sums $\sigma(f, P, \xi)$ converge to a limit, $\int_z^b f(x)\,dx$, as the diameter $\|P\|$ of the partitions converges to zero. What does this mean in terms of an ϵ–δ criterion? Well, we need

$$\exists I \in \mathbb{R} \text{ so that } \forall \epsilon > 0\ \exists \delta > 0 : \forall (P, \xi) \text{ with } \|P\| < \delta \text{ we have } |I - \sigma(f, P, \xi)| < \epsilon.$$

Note that this criterion is a little more complicated than the definition of the limit of a sequence of real numbers. The reason is that while the elements, a_n of a sequence are indexed by an integer, n, there is no way to give an ordering or indexing of the set of all tagged partitions of an interval. The best we can do is require $\|P\| < \delta$.

- You will notice in the MATH 5302 notes that most of the discussion is devoted to results of Darboux. In the mid 1870's Darboux tried to convince the French mathematics community of the need for more rigor in analysis. He devised many counter-examples which showed the danger of relying too heavily on intuition. His major contribution to Riemann's theory was to introduce the notions of upper and lower Darboux sums. Given a bounded function, $f : [a, b] \to \mathbb{R}$ and a partition, $P = \{x_0, x_1, \cdots x_n\}$ for $[a, b]$ he defines lower and upper Darboux sums

$$L(f, P, [a, b]) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \qquad \text{and} \qquad U(f, P, [a, b]) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \tag{4}$$

where m_i and M_i are the inf and sup of f on $[x_{i-1}, x_i]$. The lower and upper Darboux integrals are then

$$L(f, [a, b]) = \sup_P L(f, P, [a, b]) \qquad \text{and} \qquad U(f, [a, b]) = \inf_P U(f, P, [a, b]), \tag{5}$$

where the inf and sup are taken over all partitions P of $[a, b]$. Darboux's big result is that

Theorem 2.1. *f is Riemann integrable (RI) if and only iff $U(f, [a, b]) = L(f, [a, b])$.*

Here is a very useful criterion.

Lemma 2.2. *A bounded $f : [a, b] \to \mathbb{R}$ is RI if and only if for $\forall \epsilon > 0\ \exists P$:*

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon. \tag{6}$$

Problem 1. Prove Lemma 2.2. You may use Theorem 2.1.

Darboux also proves the following results. These results are all proved in the MATH 5302 notes.

Theorem 2.3. *Darboux also proves the following results, all of which are proved in the MATH 5302 notes.*

1. *Every continuous function $f : [a, b] \to \mathbb{R}$ is RI.*
2. *If $f : [a, b] \to \mathbb{R}$ is bounded with only a finite number of discontinuities then f is RI.*
3. *If $f : [a, b] \to \mathbb{R}$ is monotone and bounded then f is RI.*
4. *Let $f : [a, b] \to \mathbb{R}$ be RI. Then $F(x) = \int_a^x f(t)\,dx$ exists and is uniformly continuous.*
5. *(FTC II) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $F(x) = \int_a^x f(t)\,dx$ is continuously differentiable and $F'(x) = f(x)$.*
6. *(FTC I) Let $F : [a, b] \to \mathbb{R}$ be differentiable and suppose F' is RI. Then $\int_a^b F'(x)\,dx = F(b) - F(a)$.*

- A modern statement and proof of FTC I and II, referred to as the "fundamental theorem of integral calculus", was given in an 1880 paper of du Bois-Reymond in *Mathematische Annalen*. Hardy's *A Course of Pure Mathematics* (1914) has the first mention of the phrase the "fundamental theorem of calculus", omitting the adjective "integral". Richard Courant's 1934 text *Differential and Integral Calculus* has a section devoted to the FTC, which looks much like we teach it today.[1]

3 Characterizations of Riemann Integrability

Can we find an easy-to-check "if-and-only-if" criterion that guarantees that a bounded function is RI? We already know that functions with a finite number of discontinuities and functions that are monotone are RI. But there are examples of worse functions than these that are still RI. These results motivated Riemann and those that came after him to search for pathological counterexamples that oscillate a lot or have a lot of discontinuities.

Definition 3.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded and let $I \subset [a, b]$ be an interval. The* oscillation of f over I *is*

$$\omega(f; I) \;=\; \sup_I f - \inf_I f. \tag{7}$$

The oscillation of f at a point $c \in [a, b]$ *is*

$$\omega(f; c) \;=\; \inf_{I \ni c} \omega(f; I) \tag{8}$$

where the inf is taken over all intervals containing c.

Problem 2. Show that if f is not continuous at c then $\omega(f; c) > 0$.

Problem 3. Suppose that $\liminf_{x\to c} f(x) \le f(c) \le \limsup_{x\to c} f(x)$. Show that $\omega(f; c) = \limsup_{x\to c} f(x) - \liminf_{x\to c} f(x)$. Hence show that if f is continuous at c then $\omega(f; c) = 0$.

Taken together we have

Proposition 3.2. *f is continuous at c if and only if $\omega(f; c) = 0$.*

Definition 3.3. *The* outer Jordan content *of a susbet $S \subset \mathbb{R}$ is*

$$\mu_J^*(S) \;=\; \inf_{C \in \mathcal{C}(S)} \ell(C) \tag{9}$$

where $\mathcal{C}(S)$ is the set of all finite *covers of S by intervals and $\ell(C)$ is the sum of the lengths of the intervals that make up the cover C.*

[1] I have my father's 1948 reprint of this book, which he bought in 1957!

Problem 4*.* How does the definition of Jordan outer measure compare to Lebesgue outer measure? Based on this similarity, can you guess the definition of Jordan inner measure, $\mu_{*,J}$? A set is Jordan measurable if the inner and outer Jordan contents are the set are equal. Show that the set of rational numbers in $[0, 1]$ is not Jordan measurable? This example shows that countable unions of Jordan measurable sets may not be Jordan measurable. The also example explains why Lebesgue required that a countable union of Lebesgue measurable sets is Lebesgue measurable.

Hankel proved the following result which characterizes which functions are Riemann integrable.

Theorem 3.4. *Let $f : [a, b] \to \mathbb{R}$ be bounded and for any $\sigma > 0$ let*

$$S(f, \sigma) = \{x \in [a, b] \,:\, \omega(f; x) \geq \sigma\} \tag{10}$$

Then f is RI if and only if for all $\sigma > 0$, $\mu^*_J(S(f, \sigma)) = 0$.

The set $S(f, \sigma)$ is the set of points on which the oscillation of f exceeds σ. For f to be RI we need this set to be small in an appropriate sense. Specifically we need its outer Jordan content to be zero.

Problem 5*.* Show that if $\exists\sigma > 0$ with $\mu^*_J(S(f, \sigma)) > 0$ then f is not RI. To do so we can use Lemma 2.2. Given a partition, P, split it up into two families of subintervals, those for which contain points in $\mu^*_J(S(f, \sigma))$ and those that don't. Follow your nose.

Lebesgue and Vitali used Hankel's theorem 3.4 to establish the following criterion for Lebesgue measurability.

Theorem 3.5. *Let $f : [a, b] \to \mathbb{R}$ be bounded and let D be the set of discontinuities of f. Then f is RI if and only if the Lebesgue measure of D is zero.*

Problem 6*.* Prove Theorem 3.5. To do so show that $S(f, \sigma)$ is closed, the oscillation of f is nonzero at points where f is discontinuous, and then apply Hankel's theorem 3.4.

Problem 7*.* Riemann constructed a bounded function that is discontinuous at all points of the form $x = \frac{p}{2q} \in \mathbb{Q}$, where p and q are coprime. Because this set of discontinuities has Lebesgue measure zero f is RI. Let $y = S(x)$ be the sawtooth function given in Fig 7. Then Riemann's function is

$$f(x) \;=\; \sum_{n=1}^{\infty} \frac{S(nx)}{n^2}. \tag{11}$$

Show that

1. the series defining f converges uniformly
2. f bounded
3. f is periodic with period 1
4. f has the set of discontinuities described above

For fun you might like to try and make a picture of the graph of f using Matlab.

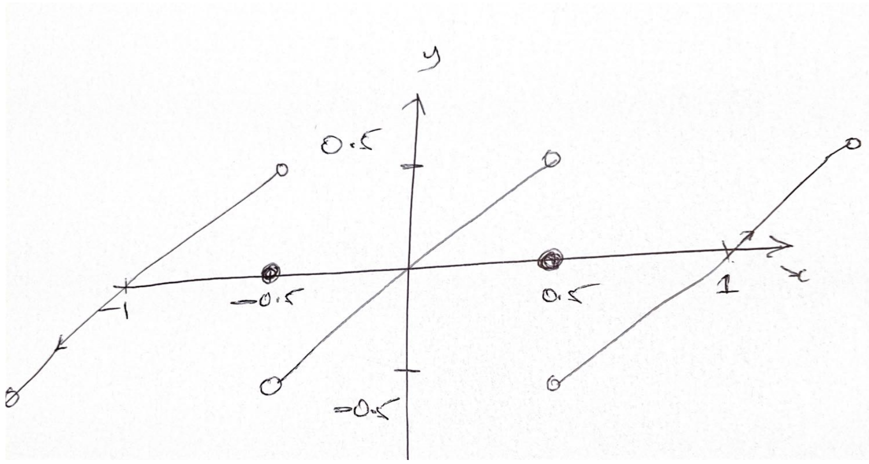

Figure 1: The Sawtooth Function

4 Short comings of the Riemann Integral

4.1 FTC

The last result in Theorem 2.3 gives a strong argument for Riemann's theory. You do not have to assume that $f = F'$ is continuous for the FTC to hold. You just need f to be RI. However this same result also leads to the ultimate downfall of the Riemann integral!

The problem is that there are functions F that are differentiable with a bounded derivative for which F' is *not* Riemann integrable. Therefore, Riemann's notions of differentiability and integrability are not completely reversible. Indeed in 1902 Lebesgue stated (quoted from [2, page 272]

> It is known that there are derivatives which are not integrable, if one accepts Riemann's definition of the integral; the kind of integration defined by Riemann does not allow in all cases to solve the fundamental problem of calculus: Find a function with a given derivative. It thus seems natural to search for a definition of the integral which make integration the inverse operation of differentiation in as large a range as possible.

Dini motivated the search for such non-RI functions with the following result.

Theorem 4.1. *Suppose $f : [a, b] \to \mathbb{R}$ is a non-constant, differentiable function so that f' is bounded and $f' = 0$ on a dense subset of $[a, b]$. Then f' is not RI.*

Problem 8. To prove Dini's Theorem assume f' is RI and obtain a contradiction using FTC I in Theorem 2.3. Recall: A subset A of $[a, b]$ is *dense* if every open interval in $[a, b]$ contains at least one element of A.

Although it was many years before such an example was found, Dini's idea led to some cool pathological functions. We will discuss two of them: The Devil's Staircase and the Smith Volterra Cantor (SVC) function.

Example 4.2 (The Devil's Staircase). Let C be the middle thirds cantor set. (See MATH 6301 lecture notes.) Define $F : [0, 1] \to [0, 1]$ as follows. If $x \in C$ then the base 3 expansion of x has the form $x = 0._3 a_1 a_2 \cdots$, where each $a_j \in \{0, 2\}$. Define $F(x) = x = 0._2 b_1 b_2 \cdots$, where $b_j = a_j/2$.

Problem 9. Show that

1. $F(1/3) = F(2/3) = 1/2$

2. If $x_1, x_2 \in C$ with $x_1 < x_2$ then $F(x_1) \leq F(x_2)$ with equality holding if and only if x_1 and x_2 are the endpoints of one of the "middle third" open intervals removed in the construction of C.

For $x \notin C$ we know that x lies in a removed open interval. Define $F(x)$ to be the common value of F on the endpoints of this open interval. For example, for $x \in (1/3, 2/3)$ we define $F(x) = 1/2$.

Problem 10. Show that

1. $F : [0, 1] \to [0, 1]$ is continuous.
2. $F(C) = [0, 1]$, that is every element $y \in [01]$ is of the form $y = F(x)$ for some $x \in C$
3. F is increasing (well, we did most of that!)
4. Try to graph F (in Matlab or by hand). It looks like a staircase with lots of horizontal steps on each of the removed middle third intervals.
5. The length of the set of horizontal steps is 1. (This statement is a little vague: can you state it more rigorously?)
6. Between any two steps there are an infinite number of other steps (hence the adjective devil!)
7. For almost all $x \in [0, 1]$, F'(x) exists and equals zero
8. Is F differentiable at all points of $[0, 1]$? Why or why not?
9. Looking ahead: Use Theorem 5.4 below to show that F is not absolutely continuous, i.e. FTC I does not hold for F.

The Devil's Staircase function suggests that if we want FTC I to hold we need F to be differentiable everywhere. However, the SVC function is differentiable everywhere on $[0, 1]$ but FTC I still does not hold. (By Theorem 2.3, for such functions F' cannot be RI.)

The SVC function is defined in terms of the SVC set.

Example 4.3 (The Smith-Volterra-Cantor (SVC) set). The SVC set is constructed exactly the same way as we constructed the middle-thirds Cantor set except that at each stage we remove from the middle of each remaining interval an open interval whose length is 1/4 (rather than 1/3) of given remianing interval.

Problem 11. Show that the SVC set has the following properties.

1. The Lebesgue measure of SVC is $1/2$
2. The SVC set is nowhere dense (contains no intervals), just like the middle third Cantor set
3. Every point in SVC is a limit point of the set of endpoints of the removed intervals.
4. The inner and outer Jordan content of SVC are not equal. (This fact motivated Lebesgue's development of Lebesgue measure.)

The second ingredient in the definition of the SVC function is a well known oscillatory function.

Problem 12. Let

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{12}$$

Show that g is differentiable, g' is bounded, $\omega(g'; x) = 0$ for all $x \neq 0$ and $\omega(g'; 0) = 2$. Hence show that g' is RI.

To get a function whose derivative is not RI we need to transplant the oscillatory behaviour of g' at zero to every point of the SVC set.

Example 4.4 (SVC function)*.* The SVC function is given by $V(x) = \sum_{n=1}^{\infty} f_n(x)$ where the functions f_n are constructed from the function g above as follows. Here we quote from the wikipedia page on the Volterra function (which I have slightly modified to suit my purposes):

> The construction of V begins by determining the largest value of x in the interval $[0, 1/8]$ for which $g'(x) = 0$. Once this value (say x_0) is determined, extend the function to the right with a constant value of $g(x_0)$ up to and including the point $1/8$. Once this is done, a mirror image of the function can be created starting at the point $1/4$ and extending left towards 0. This function will be defined to be 0 outside of the interval $[0, 1/4]$. We then translate this function to the interval $[3/8, 5/8]$ so that the resulting function, which we call f_1, is nonzero only on the middle interval of the complement of the SVC set. To construct f_2, g' is then considered on the smaller interval $[0, 1/32]$, truncated at the last place the derivative is zero, extended, and mirrored the same way as before, and two translated copies of the resulting function are added to produce the function f_2. Continuing in this fashion, we construct the sequence f_n.

Problem 13*.* Show that

1. If $x \in$ SVC, then $V(x) = 0$ and $|V(x) - V(y)| \leq (y - x)^2$. Hence by the definition of the derivative $V'(x) = 0$.
2. If $x \notin$ SVC, then there is a neighbourhood of x on which only one of the f_n is nonzero.
3. Hence $V'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for all $x \in [0, 1]$. So V is differentiable on $[0, 1]$ and V' is bounded.
4. At any endpoint x_* of a removed interval $V' = 0$ but there are points arbitrarily close to x_* at which $V' = \pm 1$.
5. Using Item (3) of Problem 11 show that $\omega(V'; x) = 2$ for every $x \in$SVC.
6. Hence show that V' is not RI.
7. However, V' is a Lebesgue mesurable function
8. Better still, V' is Lebesgue integrable and Lebesgue's Version of FTC II holds for V. **Hint:** Looking ahead to Theorem 5.4, it is enough to show V is absolutely continuous. I don't know how hard this is to verify?

4.2 Term-by-term integration

Let $f_n, f : [a, b] \to \mathbb{R}$. We recall that $f_n \to f$ *converges uniformly* if $\forall \epsilon > 0\ \exists N = N(\epsilon)$ so that $\forall n > N$ and $\forall x \in [a, b]\ |f_n(x) - f(x)| < \epsilon$. The point is that in this definition N is independent of x. If we need to assume that N depends on x as well as ϵ then we say the convergence is only *pointwise*. Also recall that a series converges (pointwise or uniformly) if the sequence of partial sums converges.

Weiestrass (1860's) showed that if $f_n; [a,b] \to \mathbb{R}$ is a sequence of RI functions for which $f(x) := \sum_{n=1}^{\infty} f_n(x)$ converges *uniformly* on $[a,b]$ then f can be integrated term-by-term:

$$\int_a^b f(x)\,dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx. \tag{13}$$

This result is extremely useful in Fourier series. However, there are cases of interest where we can swap sum and integral without having to assume uniform convergence.

Problem 14. If you assume that the sum, f of the series is RI, show that the problem of determining whether a series can be integrated term by term is equivalent to the following problem:

> Suppose that f_n is a sequence of RI functions so that $f_n(x) \to 0$ for all $x \in [a,b]$ (pointwise convergence). Under what conditions does $\int_a^b f_n(x)\,dx \to 0$?

Incidentally, the following example shows that the sum, f, of a series may not be RI even if all the f_n are.

Problem 15. Define

$$f_n(x) = \begin{cases} 1 & x = p/q, \text{with } q < n \\ 0 & \text{otherwise}. \end{cases} \tag{14}$$

Show that each f_n is RI but that $f(x) := \lim_{n\to\infty}$ is not RI.

The following examples are illustrative. They were first found between the 1870's and 1890's.

Definition 4.5. *We say that a sequence of functions* $\{f_n\}_{n=1}^{\infty}$ *on* $[a,b]$ *is* uniformly bounded *if* $\exists M$ *so that* $|f_n(x)| < M$ *for all* $x \in [a,b]$ *and all* n.

Problem 16 (**Pointwise convergence is not sufficient**). Let $f_n(x) = nxe^{-nx^2}$. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$ but that $\int_0^1 f_n(x)\,dx \to \frac{1}{2}$ as $n \to \infty$. Also show that the sequence of functions f_n is not uniformly bounded.

Problem 17. [**Uniform convergence is not necessary**] Let $g_n(x) = \frac{nx}{1+n^2x^2}$. Show that the sequence of functions g_n is bounded. Show that $g_n \to 0$ pointwise but not uniformly on $[0,1]$ and that $\int_0^1 g_n(x)\,dx \to 0$ as $n \to \infty$.

Problem 18. [**Uniform convergence is not necessary**] Let $h_n(x) = \frac{n^2x}{1+n^3x^2}$. Show that the sequence of functions h_n is not bounded. Show that $h_n \to 0$ pointwise but not uniformly on $[0,1]$ and that $\int_0^1 h_n(x)\,dx \to 0$ as $n \to \infty$.

Arzela proved a theorem (see below0 which explains Problem 17. To explain Problem 18 we must appeal to Lebesgue's Dominated Convergence Theorem (see MATH 6301 lecture notes).

Theorem 4.6 (Arzela's BCT for RI). *Suppose that* $f_n, f : [a,b] \to \mathbb{R}$ *are RI, that* $f_n \to f$ *pointwise, and that* $\{f_n\}_{n=1}^{\infty}$ *is uniformly bounded. Then* $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ *as* $n \to \infty$.

5 Lebesgue overcomes the short comings of Riemann

The point of our course is to develop Lebesgue's theory. Here we briefly summarize some of the ways it helps.

Theorem 5.1 (Lebesgue's FTC II). *Suppose that f is a Lebesgue integrable function on $[a, b]$. Then*

$$F(x) = \int_a^x f(t)\,dt. \tag{15}$$

is differentiable almost everywhere and

$$F' = f \qquad \text{almost everywhere.} \tag{16}$$

To obtain FTC I we need a new notion of continuity.

Definition 5.2. *A function $f : [a, b] \to \mathbb{R}$ is* absolutely continuous *if $\forall \epsilon > 0\ \exists \delta > 0$ so that whenever we have $a \le x_1 < y_1 \le x_2 < y_2 \le \cdots \le x_m < y_m \le b$ with $\sum_{i=1}^{m} (y_i - x_i) < \delta$ then $\sum_{i=1}^{m} |f(y_i) - f(x_i)| < \epsilon$.*

Absolutely continuous functions are continuous and are of bounded variation, which intuitively means that they do not oscillate too much. For continuous f we know that $F(x) = \int_a^x f(t)\,dt$ is continuously differentiable. But what can we say about F if we only assume that f is Lebesgue integrable?

Theorem 5.3. *If f is Lebesgue integrable on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is absolutely continuous.*

Theorem 5.4 (Lebesgue's FTC I). *Suppose that F is an absolutely continuous function on $[a, b]$. Then F is differentiable almost everywhere, $f = F'$ is Lebesgue integrable on $[a, b]$, and*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{17}$$

Combining Lebesgue's FTC I and II we see that with the Lebesgue integrals the notions of differentiability and integrability *are* completely reversible. In summary, we have the following result that was first proved by Vitali in 1905. In his paper Vitali also introduced the term "absolutely continuous", though this concept had been used by others since 1884.

Theorem 5.5. *A function F is of the form $F(x) = \int_a^x f(t)\,dt$ for some Lebesgue integrable function, f if and only if F is absolutely continuous.*

Finally, Lebesgue's theory also solves the problem of term-by-term integration. To see how he did it, takes a look in the MATH 6301 lecture notes at the material the Monotone Convergence Theorem, Fatou's Lemma, and my all time favorite, the Lebesgue Dominated Convergece Theorem. Significantly, you will find that these results rely on the countable additivity property of Lebesgue measure, that is that the Lebesgue measure of a *countable* union of disjoint sets equals the infinite sum of the Lebesgue measures of these sets.

References

[1] D. BRESSOUD, *A Radical Approach to Lebesgue's Theory of Integration*, MAA Textbooks, Cambridge University Press, 2008.

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[3] T. HAWKINS, *Lebesgue's Theory of Integration. Its Origins and Development.*, University of Wisconsin Press, 1970.