Math 6301 Homework 9

John Zweck

1. **J**.6C: 19
2. **J**.6C: 16
3. Let $f_n(x) = nxe^{-nx^2}$. Show that the conclusion of the LDCT fails when integrating f_n over the interval $[0, 1]$. Why does the hypothesis of the LDCT fail? Hint: Calculate $\sup_{n\geq 1} f$.
4. Show that if $f \in L^1$ then for all $\epsilon > 0$ there is a simple function s so that $\int |f - s| d\lambda < \epsilon$.
5. (Riemann-Lebesgue Lemma) Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \cos(nx)\, dx = 0.$$

   Hint: Use the result in the previous problem to show that it is sufficient to prove the result for a simple function. Then use the definition of a simple function to show that it is sufficient to prove it for the characteristic function of a measurable set. Then use the approximation theorem **M9** to show that if enough to prove the result whe f is the characteristic function of a closed interval.

6. Let $f \in L^1(\mathbb{R})$. Show that $f_t(x) := f(x + t) \in L^1(\mathbb{R})$ and $\int f_t d\lambda = \int f d\lambda$.

Additional Problems

1. Prove Lecture 15, Theorem 2 on Improper Riemann Integrals
2. **J**.6B: 11
3. **J**.6B: 12
4. **J**.6G: 43
5. Let $f_n(x) = (n + 1)x^n$ on $E = [0, 1]$. Show that for this sequence equality does not hold in Fatou's Lemma.
6. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & x \in [0,1] \sim \mathbb{Q} \\ 1 & x \in [0,1] \cap \mathbb{Q} \end{cases}$$

   Find the Lebesgue integral of f. Is f Riemann integrable?

7. Show that the conditions of the LDCT are sufficient but not necessary by considering the sequence $f_n = n\chi_{[\frac{1}{n}, \frac{n+1}{n^2}]}$.
8. Let $f_n(x) = nx^{n-1} - (n + 1)x^n$ on $[0, 1]$. Show that

$$\int_0^1 \left(\sum_{n=1}^{\infty} f_n(x) \right) dx \neq \sum_{n=1}^{\infty} \left(\int_0^1 f_n(x)\, dx \right)$$

and

$$\sum_{n=1}^{\infty}\left(\int_0^1 |f_n(x)|\,dx\right)=\infty.$$

9. **J**.6B: 11
10. **A**.3B: 5
11. **A**.3B: 7
12. **A**.3B: 12
13. **A**.3B: 16