Math 6301 Homework 3

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For questions (1)-(3) the only results you may use about Lebesgue measure are the properties of the Lebesgue measure of special rectangles and special polygons, as well as the definition of Lebesgue measure of an open set.

1. **A**.1B: 1

2. **A**.1B: 4

3. **J**.2: 3.

   Hint: Show that $G \sim P$ is a non-empty open set. To do so, you may need to use the **Connectedness Property** of $\mathbb{R}^n$, which states that the only subsets of $\mathbf{R}^n$ that are both open and closed are the empty set and $\mathbb{R}^n$ itself.

4. **J**.2: 4a

5. **J**.2: 12

6. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and suppose that $f(x) > 0$ for all $x \in [a, b]$. Let $G_{(a,b)}$ be the open subset of $\mathbb{R}^2$ defined by

$$G_{(a,b)} = \{(x, y) \in \mathbb{R}^2 \mid a < x < b \text{ and } 0 < y < f(x)\}.$$

   (a) Let P by any partition of $[a, b]$. Prove that

$$L(f, P, [a, b]) \leq \lambda(G_{(a,b)}) \leq U(f, P, [a, b])). \qquad (1)$$

   Carefully justify each step in your calculation using definitions and properties of the Lebesgue measure of special rectangles, special polygons, and open sets. [Here $L(f, P, [a, b]))$ and $U(f, P, [a, b]))$ are lower and upper Darboux sums.]

   (b) Use (1) to prove that $\lambda(G_{(a,b)}) = \int_a^b f$.

Additional Problem [Not not hand in]

1. **A**.1B: 5

2. **J**.2: 4b

3. **J**.2: 6

4. **J**.2: 7

5. **J**.2: 9

6. **J**.2: 10

7. Use the following deep theorem of Banach and Tarski to show that the four conditions below cannot all hold.

Theorem 1. *The unit ball in $\mathbb{R}^3$ can be decomposed into a finite number of pieces which may be reassembled, using only translation and rotation, to form two disjoint copies of the unit ball.*

(a) Every bounded set, E, in $\mathbb{R}^3$ has a volume, $\mu(E)$, with $\mu(E) \geq 0$.

(b) Let E be a bounded set in $\mathbb{R}^3$ and let F be a rotation and translation of E. Then $\mu(F) = \mu(E)$.

(c) If E is a cube of side length a then $\mu(E) = a^3$.

(d) If E_1 and E_2 are disjoint bounded sets in $\mathbb{R}^3$, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

8. Using only the definitions and properties of Lebesgue measure for special rectangles, special polygons, open sets, and compact sets can you prove the following results? If so, provide a proof. If not explain why. In these problems $A = [-2, 2] \times [-2, 2]$, B is open unit disc, and $C = [-1/2, 1/2] \times [-1/2, 1/2]$.

(a) $\lambda((0, 1)) = \lambda([0, 1])$

(b) $\lambda((0, 1]) = \lambda([0, 1])$

(c) $\lambda(A) > \lambda(B)$

(d) $\lambda(A^\circ) > \lambda(B)$

(e) $\lambda(C) < \lambda(B)$

(f) $\lambda(C^\circ) < \lambda(B)$

(g) $\lambda(C) < \lambda(\text{closure}(B))$

(h) $\lambda(C^\circ) < \lambda(\text{closure}(B))$