Math 6301 Homework 2

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Notation: **A**.1B means Axler Chapter 1. Section B.

1. **A**.1A: 1
2. **A**.1A: 2
3. **A**.1A: 4
4. **A**.1A: 11
5. **A**.1B: 3

Additional Problem [Not not hand in]

1. **A**.1A: 3
2. **A**.1B: 5
3. **J**.2.8
4. Suppose that the three desirable properties of area (Lecture 5, page 5) hold.
   (a) Let A be a bounded set in $\mathbb{R}^2$ and $B \subseteq A$. Show that $\mu(A) \geq \mu(B)$. [Hint: $A = B \cup (A \backslash B)$.]
   (b) Let A be a non-empty bounded open set in $\mathbb{R}^2$. Show that $\mu(A) > 0$.
5. Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $\lambda \in \mathbb{R}$. Prove that λf is Riemann integrable and that
$$\int_a^b \lambda f(x)\, dx \;=\; \lambda \int_a^b f(x)\, dx.$$
6. Let $f : [a, b] \to \mathbb{R}$ be bounded. Show that f is Riemann integrable if and only if $\forall \epsilon > 0$ there exists a partition, P, of $[a, b]$ with $S(f, P) - s(f, P) < \epsilon$.
7. Let $a < c < b$. If f is Riemannian integrable on $[a, c]$ and on $[c, b]$, then f is Riemannian integrable on $[a, b]$. Hint: Use result in previous problem.
8. Construct a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ with the following properties:
   (a) $f_n(x) \to f(x)$ pointwise for all $x \in [0, 1]$,
   (b) Each f_n is Riemann integrable, and
   (c) f is not Riemann integrable.
9. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. Suppose that for all $a, b \in [0, 1]$ with $a < b$ there exists a real number $c \in (a, b)$ so that $f(c) = 0$. Show that $\int_0^1 f(x)\, dx = 0$. Must $f \equiv 0$ be identically zero? What if f is continuous?
10. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $\int_a^c f(x)\, dx = 0$ for all $c \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

11. If $f : [a, b] \to \mathbb{R}$ is bounded and f^2 is Riemann integrable must f also be Riemann integrable?

12. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $c \in [a, b]$. Let C be a real number with $C \neq f(c)$. Define g by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq c \\ C & \text{if } x = c. \end{cases}$$

Prove that g is Riemann integrable on $[a, b]$.

13. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\int_a^b f(x)g(x)\,dx = 0$ for every continuous function $g : [a, b] \to \mathbb{R}$. Show that $f \equiv 0$ on $[a, b]$.