M4355 S20 E II SOLUTIONS

1 (a) These two matrices cannot be similar. Similar matrices have the same trace. But

$$\text{Trace } [T]_\beta = 1+5+9 = 15$$

$$\text{Trace } [T]_{\beta'} = 2+10+18 = 30.$$

(b) $[T]_{\beta'}$ is obtained from $[T]_\beta$ in 2 steps. First permute rows

~~R2 ← R1~~
~~R3 ←~~

which is implemented by multiplication on left by an elementary matrix:

$$\begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad (1)$$

This moves R1 to R2, R2 to R3, R3 to R1.

Second permute cols, moving C1 to C2, C2 to C3, C3 to C1.

2

Since the pattern is the same, this is implemented by the transpose of the matrix in 1 being multiplied on right

i.e.

$$\begin{pmatrix} 9 & 7 & 8 \\ 3 & 1 & 2 \\ 6 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (2)$$

Finally notice that since

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\cancel{PP = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} =}$$

$$PP^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = I_{3\times 3}$$

So $\quad P^T = P^{-1}$

So $\quad [T]_{\beta'} = P [T]_{\beta} P^{-1} \quad$ by 1 + 2

For example could chose $\mathcal{B}$ to be standard basis and $\mathcal{B}'$ to be basis given by columns of P^T.

ie $\mathcal{B}' = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Then the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is

$$P = \left([\vec{e}_1]_{\mathcal{B}'} \; [\vec{e}_2]_{\mathcal{B}'} \; [\vec{e}_3]_{\mathcal{B}'} \right)$$

and

$$[\vec{e}_1]_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ as } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[\vec{e}_2]_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ as } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[\vec{e}_3]_{\mathcal{B}'} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ as } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

This gives the P we found above ✓

(c) $\mathcal{B}$ must be ONB as only get

$$[T^*]_{\mathcal{B}} = \left(\overline{[T]_{\mathcal{B}}}\right)^T = ([T]_{\mathcal{B}})^T$$

OUR EX

for an ONB

(2) Notice

- $$AA^T = \frac{1}{9}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I = A^TA$$

- $$B^2 = \frac{1}{9}\begin{pmatrix} 2 & 2 & -1 \\ 0 & 3 & 0 \\ -2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \\ 0 & 3 & 0 \\ -2 & 4 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 6 & 6 & -3 \\ 0 & 9 & 0 \\ -6 & 12 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 0 & 3 & 0 \\ -2 & 4 & 1 \end{pmatrix}$$

$$= B$$

4

- $\det(B) = \frac{1}{27}\left[2\times3 - 2\times0 + -1\times-6\right] = 0$

- $C^2 = I$
  $C^T = C$
  $CC^T = I$ are all easy to check

(a) Any invertible matrix M can serve as a change of basis matrix from ~~bas~~ std basis to basis given by cols of M.

So A and C are change of basis matrices but not B as $\det B = 0$.

(b)

A reflector R has following properties.

$R^2 = I$ (R is invertible)

$R^T = R$

$RR^T = I$.

Since $A^T \neq A$ and $\det B = 0$

only candidate is C. To check it is really a reflector note that:

$C\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$ is reflection across xz-plane

(c) An $n\times n$ matrix P is a projector $\Leftrightarrow P^2=P$, $P\neq I$ (5)

~~Thus~~ In addition $\det P=0$ must hold as $R(P)\neq \mathbb{R}^n$.

Since A, C are invertible they cannot be projectors.

But $B^2=B$ so it is a projector

(d) A, C are orthogonal.

B is not as $\det P=\pm 1$ for orthogonal P.

(3) Notice $X \perp Y$ as if $\vec{x}\in X$ then

$\langle \vec{x} \mid \begin{pmatrix}1\\2\\3\end{pmatrix}\rangle = 0$ as $x_1+2x_2+3x_3=0$ on X.

Let $X=\text{Span}\{\vec{x}_1, \vec{x}_2\}$ where $\{\vec{x}_1, \vec{x}_2\}$ are ONB for X

Let $Y=\text{Span}\{\vec{y}_1\}$ $\|\vec{y}_1\|=1$.

The $P=[\vec{x}_1\ \vec{x}_2\ \vec{y}_1]\begin{pmatrix}I_{2\times 2} & 0\\ 0 & 0\end{pmatrix}\begin{bmatrix}\vec{x}_1^T\\ \vec{x}_2^T\\ \vec{y}_1^T\end{bmatrix}$

So $P = \vec{x}_1 \vec{x}_1^T + \vec{x}_2 \vec{x}_2^T$.

Next notice $\mathcal{X} = N((1,2,3))$ which has solution set

$$\vec{x} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

So $\mathcal{X} = \operatorname{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

by GS

Set $\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\vec{w}_2 = \vec{x}_2 - \langle \vec{u}_1 | \vec{x}_2 \rangle \vec{u}_1$$

$$= \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -6/5 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}$$

So

$$P = \frac{1}{5}\begin{pmatrix}-2\\1\\0\end{pmatrix}\begin{pmatrix}-2 & 1 & 0\end{pmatrix} + \frac{1}{70}\begin{pmatrix}-3\\-6\\5\end{pmatrix}\begin{pmatrix}-3 & -6 & 5\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}4 & -2 & 0\\-2 & 1 & 0\\0 & 0 & 0\end{pmatrix} + \frac{1}{70}\begin{pmatrix}9 & 18 & -15\\18 & 36 & -30\\-15 & -30 & 25\end{pmatrix}$$

$$= \frac{1}{70}\left[\begin{pmatrix}56 & -28 & 0\\-28 & 14 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}9 & 18 & -15\\18 & 36 & -30\\-15 & -30 & 25\end{pmatrix}\right]$$

$$= \frac{1}{70}\begin{pmatrix}65 & -10 & -15\\-10 & 50 & -30\\-15 & -30 & 25\end{pmatrix}$$

ALT APPROACH

P_X = The Projection onto X along Y

P_Y = The Projection onto Y along X.

We have $P_X + P_Y = I$ holding

P_Y is $\perp$ projection onto a line in direction of vector $\vec{n} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ so

$$P_Y \vec{u} = \frac{\vec{n}\,\vec{n}^T}{|\vec{n}|^2}\vec{u}$$

So $$P_Y = \frac{\vec{n}\,\vec{n}^T}{|\vec{n}|^2} = \frac{1}{14}\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}$$

So $$P_X = I - P_Y = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} - \frac{1}{14}\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{pmatrix}$$

$$P_X = \frac{1}{14}\begin{pmatrix}13 & -2 & -3\\-2 & 10 & -6\\-3 & -6 & 5\end{pmatrix}$$

COOL!

8

4 2

a $\langle A|B\rangle = Tr(A^*B)$

$$= \sum_{i,j=1}^{n} A^*_{ij} B_{ji}$$

$$= \sum_{i,j=1}^{n} \overline{A_{ji}} B_{ji}$$

So $\boxed{\langle A|B\rangle = \sum_{i,j=1}^{n} \overline{A_{ij}} B_{ij}}$

- $\langle A|A\rangle = \sum_{i,j=1}^{n} |A_{ij}|^2 \geq 0$

and $=0 \iff A_{ij}=0 \ \forall i,j$

$\iff A=0$.

- $\langle A|\alpha B + C\rangle = Tr(A^*(\alpha B + C))$

$= \alpha Tr(A^*B) + Tr(A^*C)$

as Tr is linear transfⁿ

$= \alpha\langle A|B\rangle + \langle A|C\rangle$

- $\langle B|A\rangle = Tr(B^*A) = Tr((B^*A)^T)$

$= Tr(A^T\overline{B}) = \overline{(Tr\, A^*B)}$

$= \overline{\langle A|B\rangle}$

- $\langle \alpha A + B|C\rangle = \overline{\langle C|\alpha A + B\rangle} = \overline{\alpha\langle C|A\rangle + \langle C|B\rangle}$

$= \overline{\alpha \langle C|A\rangle} + \overline{\langle C|B\rangle}$

$= \bar{\alpha}\langle A|C\rangle + \langle B|C\rangle$

b

- E_{ij} has 1 in (i,j)-entry and 0 elsewhere.

So for any matrix $A \in \mathbb{C}^{n\times n}$

$$\boxed{A = \sum_{i,j=1}^{n} A_{ij} E_{ij}}$$ Spanning. 1

- Next lets check $\langle E_{ij} | E_{kl}\rangle = 0$ if $(i,j) \neq (k,l)$ 2

and

$\langle E_{ij} | E_{ij}\rangle = 1$ 3

Taken together ~~these 3~~ 1, 2, 3 tell us

$\{E_{ij} \mid i,j = 1, \dots, n\}$ is an ONB for $\mathbb{C}^{n\times n}$

as 2, 3 say it is an ONB for its span (ie LI)

Well

$$\langle E_{ij} | E_{kl} \rangle = \mathrm{Tr}\left(E_{ij}^* E_{kl}\right)$$

$$= \mathrm{Tr}\left(\vec{e}_i \vec{e}_j^{\,*} \cdot \vec{e}_k \vec{e}_l^{\,*}\right)$$

$$= \mathrm{Tr}\Big(\underbrace{(\vec{e}_j^{\,*} \vec{e}_k)}_{1\times 1} \underbrace{\vec{e}_i \vec{e}_l^{\,*}}_{n\times n} \Big)$$

$$= (e_j^* e_k)^n \; \mathrm{Tr}\left(\vec{e}_i \vec{e}_l^{\,*}\right)$$

as $\mathrm{Tr}(\alpha A) = \alpha^n \mathrm{Tr}(A)$

$$= \delta_{jk} \left(e_l^* e_i\right) = \delta_{jk} \delta_{il}$$

as $\mathrm{Tr}(\vec{v}\,\vec{w}^T) = \vec{w}^T \vec{v}$

(c) In general if $T : U \to V$

and $\{\vec{u}_1, \dots, \vec{u}_n\}$ is ONB for U

Then for any $\vec{w} \in V$

$$T^*(\vec{w}) = \sum_{i=1}^{n} \langle T(\vec{u}_i) | \vec{w} \rangle_V \, \vec{u}_i$$

So

$$Tr^*(1) = \sum_{i,j=1}^{n} \langle Tr(E_{ij}) \mid 1 \rangle_{\mathbb{C}} E_{ij}$$

$$= \sum_{i,j=1}^{n} \overline{Tr(E_{ij})}\, E_{ij}$$

$$= \sum_{i,j=1}^{n} Tr(E_{ij})\, E_{ij}$$

$$= \sum_{i,j=1}^{n} \delta_{ij}\, E_{ij}$$

$$= \sum_{i=1}^{n} E_{ii}$$

$$= I_{n \times n}$$

As $Tr(E_{ij}) = Tr(e_i e_j^*)$
$= e_j^* e_i = \delta_{ij}$

<u>n=3:</u>

$$E_{11} + E_{22} + E_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I$$