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MATH 4355 [Spring 2020] Exam I, Feb 24th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations.** Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [12 pts] State the definitions of:

(a) What it means for a vector space to be finite dimensional

A vector space, V, is finite dimensional if it has a finite spanning set, i.e. if $\exists\ \mathcal{L} = \{\vec{v}_1, \dots \vec{v}_n\}$ so that $\forall \vec{v} \in V\ \exists\ \alpha_j \in \mathbb{R} : \vec{v} = \sum_{j=1}^{n} \alpha_j \vec{v}_j$

(b) A linearly independent set

A set of vectors $\mathcal{L} = \{\vec{v}_1, \dots \vec{v}_n\}$ in a V.S. V is L.I. if whenever $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}$ it must follow that $\alpha_1 = \dots = \alpha_n = 0$.

(c) The nullspace of a matrix

Let A be $m \times n$. $\quad \underset{m\times 1}{\vec{y}} = \underset{m\times n}{A}\, \underset{n\times 1}{\vec{x}}$

$$N(A) = \{\vec{x} \in \mathbb{R}^n / A\vec{x} = \vec{0}\}$$

(d) The rank of a matrix

The rank of a matrix A is the # of PIVOTS in a row echelon form E for A. The pivots are the 1st non-zero entries in each row of E

(2) [9 pts] Let

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 7 \end{pmatrix}.$$

(a) Find a nonsingular matrix $\mathbf{P}$ so that $\mathbf{PA} = \mathbf{E_A}$ where $\mathbf{E_A}$ is in reduced row echelon form, i.e, $\mathbf{E_A}$ is in row echelon form, all pivot entries are 1, and all entries above the pivots are 0.

$[A|I] \overset{\text{Row}}{\sim} [E_A|P]$

Rank $A = 2$

$$\left[\begin{array}{ccc|cc} 2 & 3 & 4 & 1 & 0 \\ 4 & 6 & 7 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R2 \to R2 - 2R1 \\ R1 \to \frac{1}{2} R1 \end{array}$$

$$\left[\begin{array}{ccc|cc} 1 & 3/2 & 2 & 1/2 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{array}\right] \quad R2 \to -R2$$

$$P = \begin{pmatrix} -7/2 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 3/2 & 2 & 1/2 & 0 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \quad R1 \to R1 - 2R2$$

$$\left[\begin{array}{ccc|cc} \boxed{1} & 3/2 & 0 & -7/2 & 2 \\ 0 & 0 & \boxed{1} & 2 & -1 \end{array}\right] = [E_A|P]$$

(b) Find nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ so that $\mathbf{PAQ}$ is in rank normal form.

$$\left[\begin{array}{c} E_A \\ \hline I \end{array}\right] \overset{\text{col}}{\sim} \left[\begin{array}{c} I_{r\times r} \\ Q \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 3/2 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \quad C2 \leftrightarrow C3 \quad \sim \left[\begin{array}{ccc} 1 & 0 & 3/2 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \quad \overset{C3 \to C3 - 3/2 C1}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & -3/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{c} I_{2\times 2} \\ \hline Q \end{array}\right]$$

$$Q = \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad P \text{ as above.}$$

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(3) [8 pts] Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that there is a matrix $\mathbf{A}$ so that $F(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Let $\{\vec{e}_1 \ldots \vec{e}_n\}$ be std basis for $\mathbb{R}^n$

$\{\vec{f}_1, \ldots, \vec{f}_m\}$ ——— $\mathbb{R}^m$

Write $\vec{x} = \sum_{j=1}^{n} x_j \vec{e}_j$ $\qquad$ A is $m \times n$

By linearity

$$F(\vec{x}) = F\left(\sum_{j=1}^{n} x_j \vec{e}_j\right)$$

$$= \sum_{j=1}^{n} x_j F(\vec{e}_j)$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij} \vec{f}_j$$

[where $F(\vec{e}_j) = \sum_{i=1}^{m} A_{ij} \vec{f}_j$ defines A_{ij}]

So $F(\vec{x}) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j\right) \vec{f}_j$

$$= \sum_{i=1}^{m} (A\vec{x})_i \vec{f}_j = A\vec{x}.$$

(4) [12 pts] Let $\mathcal{B}$ be a basis for an n-dimensional vector space, $\mathcal{V}$.
(a) Define the coordinate vector, $[\mathbf{v}]_{\mathcal{B}}$, of a vector $\mathbf{v} \in \mathcal{V}$.

Let $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$. Since $\mathcal{B}$ spans V $\exists\, \alpha_1, \dots \alpha_n \in \mathbb{R}$:
$\vec{v} = \sum_{j=1}^{n} \alpha_j \vec{v}_j$ (Since $\mathcal{B}$ is LI the α_j are!)

Define $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$.

(b) Let $T : \mathcal{V} \to \mathbb{R}^n$ be defined by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$. Prove that T is a linear transformation, and that T is one-to-one and onto. Justify any claims you make.

① $T(\lambda\vec{v} + \vec{w}) = [\lambda\vec{v} + \vec{w}]_{\mathcal{B}} \overset{\text{CLAIM}}{=} \lambda[\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$
$= \lambda T(\vec{v}) + T(\vec{w})$

CLAIM $[\lambda\vec{v} + \vec{w}]_{\mathcal{B}} = \lambda[\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$

PF If $\vec{v} = \sum \alpha_j \vec{v}_j$, $\vec{w} = \sum \beta_j \vec{v}_j$ Then
$\lambda\vec{v} + \vec{w} = \sum (\lambda\alpha_j + \beta_j)\vec{v}_j$

So $[\lambda\vec{v} + \vec{w}]_{\mathcal{B}} = \begin{pmatrix} \lambda\alpha_1 + \beta_1 \\ \vdots \\ \lambda\alpha_n + \beta_n \end{pmatrix} = \lambda\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$
$= \lambda[\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$

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② T is 1-1 : Suppose $T(\vec{v}) = T(\vec{w})$ MUST SHOW $\vec{v} = \vec{w}$
Well if $T(\vec{v}) = T(\vec{w})$, then $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = [\vec{w}]_{\mathcal{B}}$.
So $\vec{v} = \sum \alpha_j \vec{v}_j = \vec{w}$.

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③ T is ONTO : Let $\vec{x} \in \mathbb{R}^n$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Show $\exists\, \vec{v} \in V$: $T(\vec{v}) = \vec{x}$.
Set $\vec{v} = \sum x_j \vec{v}_j$. Then $T(\vec{v}) = [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{x}$.

(5) [12 pts] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ -4x+5y \end{pmatrix},$$

and let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$. Calculate the matrix, $[T]_{\mathcal{BB}}$, of T in this basis.

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

We have if $B = \{\vec{u}_1, \vec{u}_2\}$ Then

$$[T(\vec{u}_1) \quad T(\vec{u}_2)] = [\vec{u}_1, \vec{u}_2] [T]_B$$

$$\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [T]$$

So $$[T]_B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -14 & -16 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ -2 & -2 \end{pmatrix}$$

(6) [10 pts] Suppose that $\mathbf{E}$ is a row echelon form of a matrix $\mathbf{A}$. Prove that the range of $\mathbf{A}^T$ is the span of the non-zero rows of $\mathbf{E}$.

A is $m \times n$. $\quad \underset{n\times 1}{\vec{x}} = \underset{(n \times m)}{A^T}\underset{(m\times 1)}{\vec{y}}$

$$R(A^T) = \{ \vec{x} \in \mathbb{R}^n \,/\, \vec{x} = A^T\vec{y} \text{ for some } \vec{y} \in \mathbb{R}^m \}$$

Now $E = PA$ where P is invertible.

So $\quad A = P^{-1}E$

$\quad A^T = E^T (P^{-1})^T$

So

$$R(A^T) = \{ \vec{x} \in \mathbb{R}^n \,/\, \vec{x} = E^T (P^{-1})^T \vec{y} \text{ for some } \vec{y} \in \mathbb{R}^m \}$$

Now let $\vec{z} = (P^{-1})^T \vec{y}$.

$\vec{u}_j^T \in \mathbb{R}^{1\times n}$

$\vec{u}_j \in \mathbb{R}^{n\times 1}$

$$E = \begin{bmatrix} \vec{u}_1^T \\ | \\ \vec{u}_p^T \\ 0 \end{bmatrix} \quad m \times n$$

$$E^T = [\vec{u}_1 \ldots \vec{u}_n \,|\, 0]$$

$$\begin{aligned} \underset{\mathbb{R}^n}{\underset{\cap}{R(A^T)}} &= \{ E^T \vec{z} \,/\, \vec{z} \in \mathbb{R}^m \} \\ &= \{ z_1\vec{u}_1 + \cdots + z_p\vec{u}_p \,/\, \vec{z} \in \mathbb{R}^n \} \\ &= \{ z_1\vec{u}_1 + \cdots + z_n\vec{u}_n \,/\, \vec{z} \in \mathbb{R}^n \} \\ &= \text{Span N.Z. Rows} \end{aligned}$$

OR....

(6) ALTERNATE PROOF

CLAIM

If E is obtained from A by doing elementary row operations (ERO's)

Then $\text{Span Rows } A = \text{Span Rows } E$

Pf There are three types of ERO's.

(a) Multiply a row by a non-zero scalar

(b) Swap 2 rows

(c) Replace a row by that row plus a multiple of another row.

None of these operations change the span of the rows as the new rows are all linear combinations of the old ones.

Better This argument shows any row of E is a L.C. of rows of A. So ~~Span~~ $\text{Span Rows } E \subseteq \text{Span Rows } A$.

Since ERO's can all be undone (inverted) ~~this~~ we can also ~~do~~ obtain A from E by ERO's. So

$\text{Span Rows } A \subseteq \text{Span Rows } E$. $\square$

The result now follows from the following 2 facts

(A) $R(A^T) = \text{Span Cols of } A^T = \text{Span Rows } A$

(B) $\text{Span Rows } E = \text{Span Non-Zero Rows } E$. $\square$

(7) [12 pts] Let $\mathcal{S}$ be a linearly independent set of vectors in a vector space $\mathcal{V}$ and let $\mathrm{v} \in \mathcal{V}$. Prove that

$$\mathcal{S} \cup \{\mathrm{v}\} \text{ is linearly independent} \iff \mathrm{v} \notin \operatorname{Span} \mathcal{S}.$$

Let $\mathcal{S} = \{\vec{u}_1, \dots \vec{u}_n\}$

$\boxed{\Rightarrow}$ Use contrapositive. Show $\vec{v} \in \operatorname{Span}(\mathcal{S})$ implies $\mathcal{S} \cup (\vec{v})$ is LD.

Well $\vec{v} = \sum_{j=1}^{n} \alpha_j \vec{u}_j$

So $\sum_{j=1}^{n} \alpha_j \vec{u}_j - 1\vec{v} = \vec{0}$.

So we have a NonTrivial LC of $\{\vec{u}_1 - \vec{u}_n, \vec{v}\}$ giving $\vec{0}$. So $\mathcal{S} \cup (\vec{v})$ is LD

$\boxed{\Leftarrow}$ Suppose $\beta\vec{v} + \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n = \vec{0}$ (✱)

MUST SHOW $\alpha_1 \dots = \alpha_n = 0 = \beta$

(a) If $\beta \neq 0$ Then $\vec{v} = -\frac{1}{\beta}(\alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n) \in \operatorname{Span}(\mathcal{S})$ $\not\!\!\varnothing$ which is not true by assumption

(b) So $\beta = 0$. The (✱) says $\alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n = \vec{0}$

But $\mathcal{S}$ is LI So $\alpha_1 = \dots = \alpha_n = 0$ too

So $\vec{v} \cup \mathcal{S}$ is LI.