LAST NAME:	FIRST NAME:

1 /12	2 /9	3 /8	4 /12	5 /12	6 /10	7 /12	T /75

MATH 4355 [Spring 2020] Exam I, Feb 24th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [12 pts] State the definitions of:
(a) What it means for a vector space to be finite dimensional

(b) A linearly independent set

(c) The nullspace of a matrix

(d) The rank of a matrix

(2) [9 pts] Let

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 7 \end{pmatrix}.$$

(a) Find a nonsingular matrix $\mathbf{P}$ so that $\mathbf{PA} = \mathbf{E_A}$ where $\mathbf{E_A}$ is in reduced row echelon form, i.e, $\mathbf{E_A}$ is in row echelon form, all pivot entries are 1, and all entries above the pivots are 0.

(b) Find nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ so that $\mathbf{PAQ}$ is in rank normal form.

(3) [8 pts] Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that there is a matrix $\mathbf{A}$ so that $F(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

(4) [12 pts] Let $\mathcal{B}$ be a basis for an n-dimensional vector space, $\mathcal{V}$.
(a) Define the coordinate vector, $[\mathbf{v}]_{\mathcal{B}}$, of a vector $\mathbf{v} \in \mathcal{V}$.

(b) Let $T : \mathcal{V} \to \mathbb{R}^n$ be defined by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$. Prove that T is a linear transformation, and that T is one-to-one and onto. Justify any claims you make.

(5) [12 pts] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ -4x + 5y \end{pmatrix},$$

and let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$. Calculate the matrix, $[T]_{\mathcal{B}\mathcal{B}}$, of T in this basis.

(6) [10 pts] Suppose that $\mathbf{E}$ is a row echelon form of a matrix $\mathbf{A}$. Prove that the range of $\mathbf{A}^T$ is the span of the non-zero rows of $\mathbf{E}$.

(7) [12 pts] Let $\mathcal{S}$ be a linearly independent set of vectors in a vector space $\mathcal{V}$ and let $\mathbf{v} \in \mathcal{V}$. Prove that

$$\mathcal{S} \cup \{\mathbf{v}\} \text{ is linearly independent} \quad \Longleftrightarrow \quad \mathbf{v} \notin \operatorname{Span} \mathcal{S}.$$