NAME: SOLUTIONS

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MATH 430 (Fall 2008) Final Exam 2, Dec 12th

No calculators, books or notes! Show all work and give **complete explanations**.
This 75 minute exam is worth a total of 75 points.

(1) [20 pts]
(a) Define the spectrum of a $n \times n$ matrix.

The spectrum of an $n \times n$ matrix A is the set of distinct eigenvalues of A. λ is an eigenvalue of A if $\exists \vec{v} \neq \vec{0} : A\vec{v} = \lambda \vec{v}$

(b) Let $\mathcal{V}$ be a finite dimensional vector space and let $\mathcal{B}$ be a basis for $\mathcal{V}$. Define the matrix $[T]_{\mathcal{B}}$ of a linear transformation $T : \mathcal{V} \to \mathcal{V}$. Suppose that $\mathcal{B}'$ is another basis for $\mathcal{V}$. How, precisely, are $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ related?

Let $\mathcal{B} = \{\vec{v}_1 \dots \vec{v}_n\}$

Define $[T]_{\mathcal{B}} = ([T(\vec{v}_1)]_{\mathcal{B}}, \dots, [T(\vec{v}_n)]_{\mathcal{B}})$

where $[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ if $\vec{u} = \sum_{j=1}^{n} \alpha_j \vec{v}_j$

Let P be the $n \times n$ matrix defined by $P = ([\vec{v}_1]_{\mathcal{B}'}, \dots, [\vec{v}_n]_{\mathcal{B}'})$

Then $[T]_{\mathcal{B}'} = P[T]_{\mathcal{B}}P^{-1}$.

(c) State three properties that characterize the determinant of a square matrix.

I The determinant depends linearly on the 1st row

II If B is obtained from A by swapping 2 rows of A
Then $\det(B) = -\det(A)$

III $\det(I) = +1$

(d) Define the algebraic multiplicity and the geometric multiplicity of an eigenvalue. Which is larger? What can you conclude if all the eigenvalues of a matrix have algebraic multiplicity equal to 1?

① $\cdot$ AlgMult $(\lambda) = n$ means $p(x) = \det(A - xI)$
$= (x-\lambda)^n q(x)$ where q is a polynomial
and $q(\lambda) \neq 0$

② GeoMult$(\lambda) = \dim(N(A - \lambda I))$

③ $1 \leq$ GeoMult $(\lambda) \leq$ AlgMult(λ) $\forall \lambda \in \sigma(A)$

④ AlgMult (λ) = GeoMult (λ) $\forall \lambda \in \sigma(A)$ and so A is diagonalizable

(e) Carefully state the version of the Spectral Theorem for diagonalizable matrices that involves spectral projectors. (This result is sometimes called the Spectral Decomposition Theorem.)

Let G_j be the spectral projector onto $N(A - \lambda_j I)$ along $R(A - \lambda_j I)$, where λ_j is the jth distinct evalue of A. Then

① $A = \sum_{j=1}^{k} \lambda_j G_j$ where $\sigma(A) = \{\lambda_1, \dots \lambda_k\}$

② $I = \sum_{j=1}^{k} G_j$

③ $G_i G_j = 0$ if $i \neq j$ and $G_i^2 = G_i$. $\forall i, j$

(2) [15 pts] Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & -1 & 5 \end{pmatrix}.$$

(a) Calculate $\det(\mathbf{A})$ using row operations.

$$\begin{vmatrix} 0 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & -1 & 5 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & -13 & -1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 38 \end{vmatrix} = -38$$

(b) Calculate $\det(\mathbf{A})$ using a cofactor expansion.

$$\begin{vmatrix} 0 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & -1 & 5 \end{vmatrix} = -1 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix}$$

$$= -1(-1) + 3(-13) = -38$$

(c) Let $\mathbf{x} = [x_1, x_2, x_3]^T$ be the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is given above and $\mathbf{b} = [0, 3, -4]^T$. Use Cramer's Rule to calculate x_2.

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-39}{-38} = \frac{39}{38}$$

$$\det(A_2) = \det([A_{*1}, \vec{b}, A_{*3}])$$

$$= \begin{vmatrix} 0 & 0 & 3 \\ 1 & 3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 \\ 3 & -4 \end{vmatrix} = -39$$

(3) [17 pts] Suppose that $\mathbf{A}$ is a 3×3 matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and eigenspaces

$$\mathcal{N}(\mathbf{A} - 2\mathbf{I}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad \mathcal{N}(\mathbf{A} - 3\mathbf{I}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(a) Show that the function $f : \mathcal{R}^3 \to \mathcal{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ is positive for all $\mathbf{x} \neq 0$.

Since Geo Mult (2) = 2 and Geo Mult (3) = 1

The sum of geometric multiplies of A is $2+1=3=n$.

Hence A is diagonalizable.

In fact since the eigenvectors of A are mutually orthogonal $A = PDP^{-1} = PDP^T$ where P is orthogonal

$$P = \left(\begin{array}{cc|c} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{array} \right) = (X_1 | X_2) \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $f(\vec{x}) = \vec{x}^T P D P^T \vec{x} = (\vec{y}^T D^{1/2})(D^{1/2}\vec{y}) = \|D^{1/2}\vec{y}\|^2 > 0$

where $\vec{y} = P^T \vec{x}$.

(b) Calculate the spectral projectors $\mathbf{G}_1$ and $\mathbf{G}_2$ corresponding to λ_1 and λ_2.

$G_1 = X_1 X_1^*$ $\quad G_2 = X_2 X_2^*$ as $A = PDP^*$ is normal

~~orthogonal~~ as P is orthogonal. $P^* = P^T$, $X_j^* = X_j^T$.

So

$$G_1 = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$G_2 = I - G_1 = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

(c) Use (b) to solve the system of differential equations $\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$, with initial condition $\mathbf{u}(0) = (1, 2, 3)^T$.

$$\vec{u}(t) = e^{At} \vec{u}(0)$$

$$= \left(e^{\lambda_1 t} G_1 + e^{\lambda_2 t} G_2\right) \vec{u}(0)$$

$$= e^{2t} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + e^{3t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(4) [10 pts] Use least squares to find the best linear fit to the data $(x_i, y_i) = (1,2)$, $(3,5)$, $(5,7)$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$$

Least squares fit is $y = \alpha + \beta x$ where

$\vec{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ satisfies $A^T A \vec{x} = A^T \vec{b}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 9 \\ 9 & 35 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 14 \\ 52 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 9 & 14 \\ 9 & 35 & 52 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 3 & 9 & 14 \\ 0 & 1 & 5/4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 11/12 \\ 0 & 1 & 5/4 \end{array}\right)$$

$$y = \frac{11}{12} + \frac{5}{4} x$$

(5) [10 pts] Let $\mathcal{V}$ be the vector space that is spanned by the linearly independent functions $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$, $p_3(x) = x^3$. Find the eigenvalues of the linear transformation $\frac{d}{dx} : \mathcal{V} \to \mathcal{V}$ defined by $\frac{d}{dx}(f) = \frac{df}{dx}$. Is there a basis $\mathcal{B}$ for $\mathcal{V}$ so that $\left[\frac{d}{dx}\right]_{\mathcal{B}}$ is diagonal?

$$\left[\frac{d}{dx}\right]_\beta = \left(\left[\frac{d}{dx}(p_0)\right]_\beta, \dots, \left[\frac{d}{dx}(p_3)\right]_\beta\right)$$

where β is basis $\beta = (p_0, p_1, p_2, p_3)$

Now $\frac{d}{dx}(p_0) = 0$ So $\left[\frac{d}{dx}(p_0)\right]_\beta = \vec{0}$

$\frac{d}{dx}(p_1) = 1 = p_0$ So $\left[\frac{d}{dx}(p_1)\right]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\frac{d}{dx}(p_2) = 2x = 2p_1$ So $\left[\frac{d}{dx}(p_2)\right]_\beta = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$\frac{d}{dx}(p_3) = 3x^2 = 3p_2$ So $\left[\frac{d}{dx}(p_3)\right]_\beta = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$

$$A = \left[\frac{d}{dx}\right]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\det(A - \lambda I) = \lambda^4 \Rightarrow \lambda = 0$ is only evalue.

If $\exists \mathcal{B}$ so that $\left[\frac{d}{dx}\right]_{\mathcal{B}} = D$ is diagonal

Then $\exists Q$: $A = QDQ^{-1}$, $\{0\} = \sigma(A) = \sigma(D) \Rightarrow D = 0$

So $A = 0$ $\times$

(6) [6 pts] Prove that the columns of an $m \times n$ matrix $\mathbf{A}$ are linearly independent if and only if $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$.

Let $A = [\vec{v}_1 \ \cdots \ \vec{v}_n]$ $\quad \vec{x} = \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$

Then $A\vec{x} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n \quad (*)$

So cols of A are LI

$\iff$ Only solution to $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0}$ is $\alpha_j = 0 \ \forall j$

$\overset{(*)}{\iff}$ Only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

$\iff$ $N(A) = 0$ by defn of nullspace

THIS IS SOLUTION OF (9) ON NEXT PAGE

~~(7) [8 pts] Prove that λ is an eigenvalue of $\mathbf{A}$ if and only if $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.~~

$$R(\vec{c}\,\vec{d}^T) = \{ \vec{c}\,\underbrace{\vec{d}^T \vec{x}}_{1 \times 1} \mid \vec{x} \in \mathbb{R}^n \}$$

$$= \{ (\vec{d}^T \vec{x})\,\vec{c} \mid \vec{x} \in \mathbb{R}^n \}$$

$$= \operatorname{Span}(\vec{c}) \quad \cancel{\text{provided}} \text{ as } \vec{d} \neq \vec{0}$$

$(\text{choose } \vec{x} = \vec{d})$

which is 1D as $\vec{c} \neq \vec{0}$.

So $\dim R(\vec{c}\,\vec{d}^T) = 1 = \operatorname{Rank}(\vec{c}\,\vec{d}^T)$

(8) [8 pts] Let $\mathbf{P}$ be an orthogonal matrix. Prove that $\det(\mathbf{P}) = \pm 1$. Also, give an example of an orthogonal matrix with $\det(\mathbf{P}) = -1$.

$$PP^T = I$$

So $1 = \det(I) = \det(PP^T)$

$= \det(P)\det(P^T)$

$= [\det(P)]^2$ as $\det(P^T) = \det A$

So $\det(P) = \pm 1$

<u>EX</u> $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $P = P^T,\ P^2 = I,\ \det P = -1$

THIS IS SOLUTION OF (7) ON PREVIOUS PAGE

~~(9) [6 pts] Let $\mathbf{c}$ and $\mathbf{d}$ be two non-zero $n \times 1$ vectors. Calculate the rank of the matrix $\mathbf{cd}^T$.~~

λ is an evalue of A

$\Leftrightarrow \exists \vec{v} \neq \vec{0} : A\vec{v} = \lambda \vec{v}$

$\Leftrightarrow \exists \vec{v} \neq \vec{0} : (A - \lambda I)\vec{v} = \vec{0}$

$\Leftrightarrow N(A - \lambda I) \neq 0$

$\Leftrightarrow A - \lambda I$ is singular

$\Leftrightarrow \det(A - \lambda I) = 0$

(10) [6 pts] Let $\mathbf{T} : \mathcal{R}^n \to \mathcal{R}$ be a linear transformation. Find a vector $\mathbf{u}$ so that $\mathbf{T}(\mathbf{v}) = \mathbf{u}^T\mathbf{v}$ for all $\mathbf{v} \in \mathcal{R}^n$. Hint: Express $\mathbf{v}$ in the standard basis for $\mathcal{R}^n$.

Let $\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j$ where $\mathcal{E} = (\vec{e}_1 - \vec{e}_n)$ is std basis

So by linearity of T

$$T(\vec{v}) = \sum_{j=1}^{n} v_j (T(\vec{e}_j)) = \vec{u}^T \vec{v}$$

where $\vec{u} = \begin{bmatrix} T(\vec{e}_1) \\ | \\ T(\vec{e}_n) \end{bmatrix}$

(11) [14 pts] Let $\mathbf{A}$ be an $m \times n$ matrix with complex entries.
(a) Prove that $\mathcal{R}(\mathbf{A})^{\perp} = \mathcal{N}(\mathbf{A}^*)$.

$$\vec{x} \in R(A)^{\perp} \iff \langle \vec{x} | A\vec{y} \rangle = 0 \quad \forall \vec{y} \in \mathbb{C}^n$$

$$\iff \langle A^* \vec{x} | \vec{y} \rangle = 0 \quad \forall \vec{y} \in \mathbb{R}^n$$

by property of adjoint

$$\iff A^* \vec{x} = \vec{0}$$

$$\iff \vec{x} \in N(A^*)$$

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(c) PROOF 2

$$N(A)^{\perp} = N\left((A^*)^*\right)^{\perp} \overset{(a)}{=} \left(\left(R(A^*)\right)^{\perp}\right)^{\perp}$$

$$= R(A^*) \quad \text{as} \quad (M^{\perp})^{\perp} = M.$$

Here I am apply (a) to the matrix A^* which is OK as (a) holds for ANY matrix.

(b) Prove that $\mathcal{R}(\mathbf{A}^*) \subseteq \mathcal{N}(\mathbf{A})^{\perp}$.

Let $\vec{x} \in R(A^*)$

So $\vec{x} = A^* \vec{z}$ for some $\vec{z} \in \mathbb{R}^n$.

Let $\vec{y} \in N(A)$. Then

$$\langle \vec{x} \mid \vec{y} \rangle = \langle A^* \vec{z} \mid \vec{y} \rangle = \langle \vec{z} \mid A\vec{y} \rangle = \langle \vec{z} \mid \vec{0} \rangle = 0$$

So $\vec{x} \in N(A)^{\perp}$.

So $R(A^*) \subseteq N(A)^{\perp}$

(c) Using (a) and (b) prove that $\mathcal{R}(\mathbf{A}^*) = \mathcal{N}(\mathbf{A})^{\perp}$.

$\boxed{\text{PROOF 1}}$ By (b) and Subspace Dim. Thm it suffices to show $\dim R(A^*) = \dim N(A)^{\perp}$.

By Rank + Nullity Thm

$A^* : \mathbb{C}^m \to \mathbb{C}^n$

$A : \mathbb{C}^n \to \mathbb{C}^m$

A^* is $n \times m$

$$\begin{aligned}
\dim R(A^*) &= m - \dim N(A^*) \\
&= m - \dim R(A)^{\perp} \quad \text{by (a)} \\
&= m - (m - \dim R(A)) \quad \text{as } \dim R(A) + \dim R(A)^{\perp} = m \\
&= \dim R(A) \\
&= n - \dim N(A) \quad \text{by R+N Thm} \\
&= \dim N(A)^{\perp}.
\end{aligned}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________