NAME:

1 /20	2 /15	3 /17	4 /10	5 /10	6 /6
7 /8	8 /8	9 /6	10 /6	11 /14	T /120

MATH 430 (Fall 2008) Final Exam 2, Dec 12th

No calculators, books or notes! Show all work and give **complete explanations**.
This 75 minute exam is worth a total of 75 points.

(1) [20 pts]
(a) Define the spectrum of a $n \times n$ matrix.

(b) Let $\mathcal{V}$ be a finite dimensional vector space and let $\mathcal{B}$ be a basis for $\mathcal{V}$. Define the matrix $[T]_{\mathcal{B}}$ of a linear transformation $T : \mathcal{V} \to \mathcal{V}$. Suppose that $\mathcal{B}'$ is another basis for $\mathcal{V}$. How, precisely, are $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ related?

(c) State three properties that characterize the determinant of a square matrix.

(d) Define the algebraic multiplicity and the geometric multiplicity of an eigenvalue. Which is larger? What can you conclude if all the eigenvalues of a matrix have algebraic multiplicity equal to 1?

(e) Carefully state the version of the Spectral Theorem for diagonalizable matrices that involves spectral projectors. (This result is sometimes called the Spectral Decomposition Theorem.)

(2) [15 pts] Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & -1 & 5 \end{pmatrix}.$$

(a) Calculate $\det(\mathbf{A})$ using row operations.

(b) Calculate $\det(\mathbf{A})$ using a cofactor expansion.

(c) Let $\mathbf{x} = [x_1, x_2, x_3]^T$ be the solution of $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is given above and $\mathbf{b} = [0, 3, -4]^T$. Use Cramer's Rule to calculate x_2.

(3) [17 pts] Suppose that $\mathbf{A}$ is a 3×3 matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and eigenspaces

$$\mathcal{N}(\mathbf{A} - 2\mathbf{I}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad \mathcal{N}(\mathbf{A} - 3\mathbf{I}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(a) Show that the function $f : \mathcal{R}^3 \to \mathcal{R}$ defined by $f(\mathbf{x}) = \mathbf{x}^T \mathbf{Ax}$ is positive for all $\mathbf{x} \neq 0$.

(b) Calculate the spectral projectors $\mathbf{G}_1$ and $\mathbf{G}_2$ corresponding to λ_1 and λ_2.

(c) Use (b) to solve the system of differential equations $\frac{d\mathbf{u}}{dt} = \mathbf{Au}$, with initial condition $\mathbf{u}(0) = (1, 2, 3)^T$.

(4) [10 pts] Use least squares to find the best linear fit to the data $(x_i, y_i) = (1, 2)$, $(3, 5)$, $(5, 7)$.

(5) [10 pts] Let $\mathcal{V}$ be the vector space that is spanned by the linearly independent functions $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$, $p_3(x) = x^3$. Find the eigenvalues of the linear transformation $\frac{d}{dx} : \mathcal{V} \to \mathcal{V}$ defined by $\frac{d}{dx}(f) = \frac{df}{dx}$. Is there a basis $\mathcal{B}$ for $\mathcal{V}$ so that $\left[\frac{d}{dx}\right]_{\mathcal{B}}$ is diagonal?

(6) [6 pts] Prove that the columns of an $m \times n$ matrix $\mathbf{A}$ are linearly independent if and only if $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$.

(7) [8 pts] Prove that λ is an eigenvalue of $\mathbf{A}$ if and only if $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.

(8) [8 pts] Let $\mathbf{P}$ be an orthogonal matrix. Prove that $\det(\mathbf{P}) = \pm 1$. Also, give an example of an orthogonal matrix with $\det(\mathbf{P}) = -1$.

(9) [6 pts] Let $\mathbf{c}$ and $\mathbf{d}$ be two non-zero $n \times 1$ vectors. Calculate the rank of the matrix $\mathbf{c}\mathbf{d}^T$.

(10) [6 pts] Let $\mathbf{T} : \mathcal{R}^n \to \mathcal{R}$ be a linear transformation. Find a vector $\mathbf{u}$ so that $\mathbf{T}(\mathbf{v}) = \mathbf{u}^T\mathbf{v}$ for all $\mathbf{v} \in \mathcal{R}^n$. Hint: Express $\mathbf{v}$ in the standard basis for $\mathcal{R}^n$.

(11) [14 pts] Let $\mathbf{A}$ be an $m \times n$ matrix with complex entries.
(a) Prove that $\mathcal{R}(\mathbf{A})^{\perp} = \mathcal{N}(\mathbf{A}^*)$.

(b) Prove that $\mathcal{R}(\mathbf{A}^*) \subseteq \mathcal{N}(\mathbf{A})^\perp$.

(c) Using (a) and (b) prove that $\mathcal{R}(\mathbf{A}^*) = \mathcal{N}(\mathbf{A})^\perp$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________