NAME: SOLUTIONS

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MATH 430 (Fall 2008) Exam 2, Nov 3

No calculators, books or notes! Show all work and give **complete explanations**.
This 75 minute exam is worth a total of 75 points.

(1) [22pts]
(a) Define the orthogonal complement of a subspace $\mathcal{M}$ of an inner product space $\mathcal{V}$.

The orthogonal complement, $\mathcal{M}^\perp$, of $\mathcal{M}$ in $\mathcal{V}$ is defined by

$$\mathcal{M}^\perp = \{ \vec{v} \in \mathcal{V} \,/\, \langle \vec{v} | \vec{m} \rangle = 0 \quad \forall \vec{m} \in \mathcal{M} \}$$

(b) State the definition of a least squares solution of a linear system. ~~Under what circumstances is the least squares solution the same as that of the original linear system~~? If the least squares solution is unique, what is it?

Consider a linear system $A\vec{x} = \vec{b}$, where $\vec{x}$ is $n \times 1$ and A is $m \times n$.

Let $Q(\vec{x}) = (A\vec{x} - \vec{b})^T (A\vec{x} - \vec{b})$, so that $Q : \mathbb{R}^n \to \mathbb{R}$.

A least squares solution of $A\vec{x} = \vec{b}$ is a vector $\vec{x}$ that minimizes $Q(\vec{x})$. If we let $\vec{\varepsilon} = A\vec{x} - \vec{b}$ be the residual vector, then $Q(\vec{x}) = \vec{\varepsilon}^T \vec{\varepsilon} = \|\vec{\varepsilon}\|^2$.

So we are minimizing the length of the residual vector.

If the LSS is ~~the~~ unique its solution is given by the unique solution of the normal equations

$$A^T A \vec{x} = A^T \vec{b}.$$

So the solution is

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}$$

(c) State the Orthogonal Decomposition Theorem.

Let A be an $m \times n$ real matrix. Then

① $R(A)^{\perp} = N(A^T)$ so $\mathbb{R}^m = R(A) \oplus N(A^T)$

② $N(A)^{\perp} = R(A^T)$ so $\mathbb{R}^n = N(A) \oplus R(A^T)$

(d) Which of the following matrices are unitary, and why?

$$\mathbf{A} = \begin{pmatrix} 3i/5 & -4/5 \\ 4/5 & 3i/5 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

$$A^* A = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{-3i}{5} \end{pmatrix} \begin{pmatrix} \frac{3i}{5} & \frac{-4}{5} \\ 4/5 & \frac{3i}{5} \end{pmatrix} = \begin{pmatrix} 1 & \frac{24i}{25} \\ -\frac{24i}{25} & 1 \end{pmatrix}$$

$$\neq I$$

So A is <u>NOT</u> unitary

$$B^* B = B^T B = \begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

So B is unitary (and orthogonal)

(2) [9 pts] Let $\mathcal{B}$ be the basis for $\mathcal{R}^2$ given by $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$. Let $\mathbf{T} : \mathcal{R}^2 \to \mathcal{R}^2$ be the linear transformation given by $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$. Calculate $[\mathbf{T}]_{\mathcal{B}}$.

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 2 \end{pmatrix} + d\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

So $$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

$$= \frac{1}{3-2}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 14 \\ -7 & -10 \end{pmatrix} = [T]_{\mathcal{B}}$$

since $[T]_{\mathcal{B}} = \left(\left[T\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{B}}, \left[T\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]_{\mathcal{B}} \right)$

(3) [12 pts] Let $\mathcal{X}$ and $\mathcal{Y}$ be the subspaces of $\mathcal{R}^2$ defined by $\mathcal{X} = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and $\mathcal{Y} = \text{Span}\left\{\begin{pmatrix}1\\3\end{pmatrix}\right\}$.

(a) Show that $\mathcal{R}^2 = \mathcal{X} \oplus \mathcal{Y}$.

$B_x = \left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$, $B_y = \left\{\begin{pmatrix}1\\3\end{pmatrix}\right\}$ are bases for $\mathcal{X}, \mathcal{Y}$.

Also $B_x \cap B_Y = \emptyset$ and $B = B_x \cup B_Y = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$. So by the complementary subspace Theorem, $\mathbb{R}^2 = \mathcal{X} \oplus \mathcal{Y}$.

(b) Calculate the matrix of the projector onto $\mathcal{X}$ along $\mathcal{Y}$ with respect to the standard basis for $\mathcal{R}^2$.

$$X = \begin{pmatrix}1\\2\end{pmatrix} \qquad Y = \begin{pmatrix}1\\3\end{pmatrix}$$

$$P = (X|Y)\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}(X|Y)^{-1}$$

$$= \begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}\begin{pmatrix}3 & -1\\ -2 & +1\end{pmatrix} = \begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix}$$

(c) Let $\mathbf{v} = \begin{pmatrix}5\\7\end{pmatrix}$. Use your answer to (b) to find vectors $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$ such that $\mathbf{v} = \mathbf{x} + \mathbf{y}$.

$$\vec{x} = P\vec{v} = \begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix}\begin{pmatrix}5\\7\end{pmatrix} = \begin{pmatrix}8\\16\end{pmatrix}$$

$$\vec{y} = \vec{v} - \vec{x} = \begin{pmatrix}5\\7\end{pmatrix} - \begin{pmatrix}8\\16\end{pmatrix} = \begin{pmatrix}-3\\-9\end{pmatrix}$$

(4) [8 pts] Use the Gram-Schmidt process to find an orthonormal basis for the vector subspace $\mathcal{V}$ of $\mathcal{R}^3$ given by

$$\mathcal{V} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Let $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{v}_1 = \vec{u}_1 / \|\vec{u}_1\| = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

And $\vec{u}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\vec{w}_2 = \vec{u}_2 - \langle \vec{u}_2 | \vec{v}_1 \rangle \vec{v}_1$

$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \frac{1}{3} (2 \ 3 \ 4) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \frac{9}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\vec{v}_2 = \vec{w}_2 / \|\vec{w}_2\| = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

ONB is $\{ \vec{v}_1 \quad \vec{v}_2 \}$

(5) [8 pts] Let **A** be an invertible $n \times n$ matrix with complex entries. Prove that

$$\langle \mathbf{x} | \mathbf{y} \rangle_{\mathbf{A}} := \mathbf{x}^* \mathbf{A}^* \mathbf{A} \mathbf{y}$$

is an inner product on $\mathcal{C}^n$.

① $<\vec{x}|\vec{x}>_A = \vec{x}^* A^* A \vec{x} = (A\vec{x})^* A\vec{x}$

$= \|A\vec{x}\|^2 \geq 0.$

where $\|\vec{v}\|^2 = <\vec{v}|\vec{v}>$ and $<\vec{v}|\vec{w}>$ is the standard inner product on $\mathbb{C}^n$.

Also $<\vec{x}|\vec{x}>_A = \|A\vec{x}\|^2 = 0 \iff A\vec{x} = \vec{0}$

(since $<\cdot|\cdot>$ is an inner product)

$\iff \vec{x} = 0$ as A is invertible

② $<\vec{x}|\alpha\vec{y}> = \vec{x}^* A^* A (\alpha\vec{y}) = \alpha \vec{x}^* A^* A \vec{y}$

$= \alpha <\vec{x}|\vec{y}> \quad \forall \alpha \in \mathbb{C},\ \vec{x}, \vec{y} \in \mathbb{C}^n$

③ $<\vec{x}|\vec{y} + \vec{z}> = \vec{x}^* A^* A (\vec{y} + \vec{z}) = \vec{x}^* A^* A \vec{y} + \vec{x}^* A^* A \vec{z}$

$= <\vec{x}|\vec{y}> + <\vec{x}|\vec{z}> \quad \forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{C}^n$

④ $\overline{<\vec{y}|\vec{x}>} = \overline{\vec{y}^* A^* A \vec{x}} = \overline{(\vec{y}^* A^* A \vec{x})^T} = (\vec{y}^* A^* A \vec{x})^*$

(6) [8 pts] Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be an orthonormal basis for ~~an~~ a complex inner-product space $\mathcal{V}$.

(a) Prove that

$$\langle \mathbf{x} | \mathbf{y} \rangle = \sum_{i=1}^{n} \langle \mathbf{x} | \mathbf{v}_i \rangle \langle \mathbf{v}_i | \mathbf{y} \rangle \qquad \text{for all } \mathbf{x} \text{ and } \mathbf{y} \text{ in } \mathcal{V}.$$

$$\vec{x} = \sum_{i=1}^{n} \langle \vec{v}_i | \vec{x} \rangle \vec{v}_i , \quad \vec{y} = \sum_{j=1}^{n} \langle \vec{v}_j | \vec{y} \rangle \vec{v}_j$$

$$\langle \vec{x} | \vec{y} \rangle = \sum_{i,j=1}^{n} \langle \langle \vec{v}_i | \vec{x} \rangle \vec{v}_i \,|\, \langle \vec{v}_j | \vec{y} \rangle \vec{v}_j \rangle$$

$$= \sum_{i,j=1}^{n} \overline{\langle v_i | \vec{x} \rangle} \langle \vec{v}_j | \vec{y} \rangle \langle \vec{v}_i | \vec{v}_j \rangle$$

$$= \sum_{i,j=1}^{n} \langle \vec{x} | \vec{v}_i \rangle \langle \vec{v}_j | \vec{y} \rangle \delta_{ij}$$

$$= \sum_{i,j=1}^{n} \langle \vec{x} | \vec{v}_i \rangle \langle \vec{v}_i | \vec{y} \rangle$$

BETTER PROOF:

$$\langle \vec{x} | \vec{y} \rangle = \langle \vec{x} | \sum_{j=1}^{n} \langle \vec{v}_j | \vec{y} \rangle \vec{v}_j \rangle$$

$$= \sum_{j=1}^{n} \langle \vec{v}_j | \vec{y} \rangle \langle \vec{x} | \vec{v}_j \rangle$$

$$= \sum_{i=1}^{n} \langle x | \vec{v}_i \rangle \langle \vec{v}_i | \vec{y} \rangle$$

(b) Suppose that $\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$. Prove that $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} |\lambda_i|^2$.

$$\vec{x} = \sum_{i=1}^{n} \lambda_i \vec{v}_i = \sum_{i=1}^{n} \langle \vec{v}_i | \vec{x} \rangle \vec{v}_i , \quad \lambda_i = \langle \vec{v}_i | \vec{x} \rangle$$

So by (a)

$$\|\vec{x}\|^2 = \langle \vec{x} | \vec{x} \rangle = \sum_{i=1}^{n} \langle \vec{x} | \vec{v}_i \rangle \langle \vec{v}_i | \vec{x} \rangle$$

$$= \sum_{i=1}^{n} \overline{\langle \vec{v}_i | \vec{x} \rangle} \langle \vec{v}_i | \vec{x} \rangle$$

$$= \sum_{i=1}^{n} \overline{\lambda_i} \lambda_i = \sum_{i=1}^{n} |\lambda_i|^2.$$

(7) [8 pts] Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of an inner product space $\mathcal{V}$. Prove that $(\mathcal{M}+\mathcal{N})^\perp = \mathcal{M}^\perp \cap \mathcal{N}^\perp$.

$\boxed{(\mathcal{M}+\mathcal{N})^\perp \subseteq \mathcal{M}^\perp \cap \mathcal{N}^\perp}$

Let $\vec{x} \in (\mathcal{M}+\mathcal{N})^\perp$. (*)

To show $\vec{x} \in \mathcal{M}^\perp$, let $\vec{m} \in \mathcal{M}$.

Then $\vec{m} = \vec{m} + \vec{0} \in \mathcal{M}+\mathcal{N}$ too.

So $\langle \vec{x} | \vec{m} \rangle = 0$ by (*). So $\vec{x} \in \mathcal{M}^\perp$.

Similarly $\vec{x} \in \mathcal{N}^\perp$. So $\vec{x} \in \mathcal{M}^\perp \cap \mathcal{N}^\perp$. ✓

$\boxed{\mathcal{M}^\perp \cap \mathcal{N}^\perp \subseteq (\mathcal{M}+\mathcal{N})^\perp}$

Let $\vec{x} \in \mathcal{M}^\perp \cap \mathcal{N}^\perp$

Let $\vec{m} + \vec{n} \in \mathcal{M}+\mathcal{N}$.

Then $\langle \vec{x} | \vec{m} + \vec{n} \rangle = \langle \vec{x} | \vec{m} \rangle + \langle \vec{x} | \vec{n} \rangle = 0$ TO

as $\vec{x} \in \mathcal{M}^\perp$ and $x \in \mathcal{N}^\perp$.

So $\vec{x} \in (\mathcal{M}+\mathcal{N})^\perp$. ✓

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________