NAME: SOLUTIONS

1 /15	2 /6	3 /16	4 /10	5 /10	6 /8	7 /10	T /75

MATH 430 (Fall 2008) Exam 1, Sep 29

No calculators, books or notes! Show all work and give **complete explanations**.
This 75 minute exam is worth a total of 75 points.

(1) [15 pts]
(a) Define the nullspace and range of a matrix.

Let A be an $m \times n$ matrix. The nullspace of A is

$N(A) = \{ \vec{x} \in \mathbb{R}^n / A\vec{x} = \vec{0} \}$ and the range of A is

$$R(A) = \{ \vec{y} \in \mathbb{R}^m / \exists \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{y} \} = \{ A\vec{x} / \vec{x} \in \mathbb{R}^n \}$$

(b) State the Rank and Nullity Theorem, and illustrate what it says in the context of a well-chosen example.

THM Let A be an $m \times n$ matrix. Then

$$\dim N(A) + \dim R(A) = n$$

EX Let $A_{m\times n} = \left(\begin{array}{c|c} I_{r\times r} & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array} \right)$

Let $\vec{e}_1 - \vec{e}_n$ be standard basis for $\mathbb{R}^n$ and $\vec{f}_1 - \vec{f}_m$ be standard basis for $\mathbb{R}^m$.

Then

$R(A) = \text{Span}\{ \vec{f}_1, \dots, \vec{f}_r \}$ has dimension r

$N(A) = \text{Span}\{ \vec{e}_{r+1}, \dots, \vec{e}_n \}$ has dimension $n-r$.

(c) Define the concept of a maximal linearly independent subset of a finite dimensional vector space.

A subset $\mathcal{S} = \{\vec{v}_1, \dots, \vec{v}_n\}$ of a vector space V is a maximal linearly independent (LI) subset of V if

① $\mathcal{S}$ is a LI subset, ie If $\sum_{i=1}^{n} \alpha_i \vec{v}_i = \vec{0}$ then $\alpha_i = 0 \ \forall i$

and

② If $\tilde{\mathcal{S}}$ is any other L.I. subset of V then $\mathcal{S}$ has the same ~~#~~ or more elements than $\tilde{\mathcal{S}}$.

(2) [6 pts] Let $\mathbf{A}$ be a block matrix of the form $\mathbf{A} = \begin{pmatrix} \mathbf{B} \\ \mathbf{C} \end{pmatrix}$. Prove that $N(\mathbf{A}) = N(\mathbf{B}) \cap N(\mathbf{C})$.

First observe that $A\vec{x} = \begin{pmatrix} B \\ C \end{pmatrix} \vec{x} = \begin{pmatrix} B\vec{x} \\ C\vec{x} \end{pmatrix}$ (*)

$\boxed{N(A) \subseteq N(B) \cap N(C)}$

Let $\vec{x} \in N(A)$. Then $A\vec{x} = \begin{pmatrix} B\vec{x} \\ C\vec{x} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}$

So $B\vec{x} = \vec{0}$ and $C\vec{x} = \vec{0}$.

So $\vec{x} \in N(B)$ and $\vec{x} \in N(C)$

$\therefore$ ~~$N(A)$~~ $\vec{x} \in N(B) \cap N(C)$.

$\boxed{N(B) \cap N(C) \subseteq N(A)}$

Let $\vec{x} \in N(B) \cap N(C)$. So $\vec{x} \in N(B)$ and $\vec{x} \in N(C)$.

So $B\vec{x} = \vec{0}$ and $C\vec{x} = \vec{0}$.

(3) [16 pts] Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 9 & 7 \\ 3 & 7 & 4 \\ 8 & 18 & 14 \end{pmatrix}. \quad A \text{ is } m \times n = 4 \times 3$$

$A|I$ Find bases for the four fundamental subspaces of $\mathbf{A}$.

$$= \left(\begin{array}{ccc|cccc} 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 4 & 9 & 7 & 0 & 1 & 0 & 0 \\ 3 & 7 & 4 & 0 & 0 & 1 & 0 \\ 8 & 18 & 14 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|cccc} 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 1 & -5 & -3 & 0 & 1 & 0 \\ 0 & 2 & -10 & -8 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R2 \rightarrow R2 - 4R1 \\ R3 \rightarrow R3 - 3R1 \\ R4 \rightarrow R4 - 8R1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|cccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -5 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R3 \rightarrow R3 - R2 \\ R4 \rightarrow R4 - 2R2 \end{array}$$

A Row Echelon Form

$= (U|P)$

① $r = 2$, $R(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 7 \\ 18 \end{pmatrix} \right\}$ Basic cols of A

② $R(A^T) = \text{Span}\{(1\ 2\ 3), (0\ 1\ -5)\}$ Nonzero rows of U

③ $N(A^T) = \text{Span}\{(1, -1, 1, 0), (0, -2, 0, 1)\}$ Last $m - r = 2$ rows of P

④ $x_1 + 2x_2 + 3x_3 = 0$ $\quad x_3$ free

(4) [10 pts]
(a) Prove that if a matrix is both symmetric and skew-symmetric then it is zero.

Let A be a square matrix that is both symmetric and skew-symmetric.

Then $A^T = A$ and $A^T = -A$

So $A = A^T = -A$

So $2A = 0$
So $A = 0$

(b) *Without using matrices* prove that the composition of two linear mappings between vector spaces is linear.

Let $f : V_1 \longrightarrow V_2$ and $g : V_2 \longrightarrow V_3$

be linear. We must show that

$g \circ f : V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ is linear.

Let $\alpha \in \mathbb{R}$, $\vec{x}, \vec{y} \in V_1$. Then

$$(g \circ f)(\alpha \vec{x} + \vec{y}) = g(f(\alpha \vec{x} + \vec{y}))$$
$$= g(\alpha f(\vec{x}) + f(\vec{y})) \quad \text{as } f \text{ linear}$$
$$= \alpha g(f(\vec{x})) + g(f(\vec{y})) \quad \text{as } g \text{ linear}$$
$$= \alpha (g \circ f)(\vec{x}) + (g \circ f)(\vec{y})$$

(5) [10 pts]

(a) Let **A** be $m \times n$ and **B** be $n \times \ell$. Prove that each column of **AB** can be expressed as a linear combina of the columns of **A**. In particular, find the coefficients in these linear combinations.

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

So $$(AB)_{*j} = \sum_{k=1}^{n} B_{kj} A_{*k}$$

i.e. The jth column of AB is ~~the~~ a linear combination of the columns of A, where the ~~k-t~~ coefficient of the kth column of is given by B_{kj}.

(b) Let

$$\mathbf{A} = \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} \qquad \mathbf{B} = \underset{2\times 4}{\begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \end{pmatrix}}$$

AB is 3

Use the formula you derived in (a) to calculate the 3rd column of **AB**.

$$(AB)_{*3} = \sum_{k=1}^{2} B_{k3} A_{*k}$$

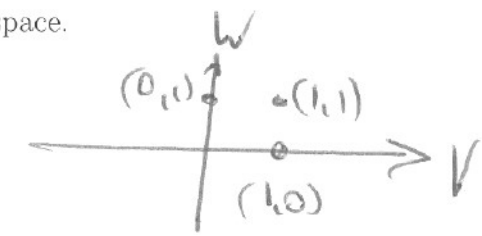
$$= B_{13} A_{*1} + B_{23} A_{*2}$$

$$= 9 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 13 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 9 + 26 \\ 27 + 52 \end{pmatrix} \qquad \begin{pmatrix} 35 \\ 7- \end{pmatrix}$$

(6) [8 pts] For each of the following statements either prove that the statement is true or give a *specific* counterexample.

(a) The union of two vector subspaces of a vector space is a vector subspace.



Let $V = \{(x,0) \in \mathbb{R}^2 / x \in \mathbb{R}\}$

$W = \{(0,y) \in \mathbb{R}^2 / y \in \mathbb{R}\}$

V, W are subspaces of $\mathbb{R}^2$.

But $V \cup W$ is NOT since it is not closed under vector addition:

$(1,0) \in V$, $(0,1) \in W$ so

$(1,0), (0,1) \in V \cup W$

but $(1,0) + (0,1) = (1,1) \notin V \cup W$.

(b) The intersection of two vector subspaces of a vector space is a vector subspace.

Let V_1, V_2 be subspaces of a vector space W

We will show $V_1 \cap V_2$ is a subspace of W

Let $\vec{x}, \vec{y} \in V_1 \cap V_2$ and $\alpha \in \mathbb{R}$

Then $\vec{x}, \vec{y} \in V_1 \Rightarrow \vec{x} + \alpha\vec{y} \in V_1$ as V_1 is a subspace of W

And $\vec{x}, \vec{y} \in V_2 \Rightarrow \vec{x} + \alpha\vec{y} \in V_2$ as V_2 is a subspace of W

(7) [10 pts] Find a basis for the vector space consisting of all 3×3 skew-symmetric matrices and prove that it is indeed a basis.

Any 3x3 skew-symmetric matrix is of the form

$$A \overset{(**)}{=} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \quad \text{where } a, b, c \in \mathbb{R}$$

Now

$$A \overset{(*)}{=} a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Let

$$B = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

The elements of B are 3x3 skew symmetric matrices. By (*) B spans the vector space of all 3x3 skew-symmetric matrices.

Suppose

$$a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: