NAME:

1 /32	2 /10	3 /12	4 /12	5 /20	6 /12	7 /10	8 /12	T /120

MATH 430 (Fall 2006) Final Exam, December 18th

Show all work and give **complete explanations** for all your answers.
This is a 120 minute exam. It is worth a total of 120 points.

(1) [32 pts]
(a) Define what it means for a square matrix to be normal. State the Spectral Theorem for normal matrices.

(b) State three equivalent conditions that guarantee that a real-symmetric matrix is positive definite.

(c) Define what it means for a matrix to be unitary. What can you say about the determinant of a unitary matrix, and why?

(d) Suppose that $\mathbf{A}$ is a real 4×4 matrix such that

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} \in \mathcal{R}(\mathbf{A}) \qquad \text{and} \qquad \begin{pmatrix} 3 \\ -1 \\ 0 \\ 2 \end{pmatrix} \in \mathcal{N}(\mathbf{A}).$$

Can $\mathbf{A}$ be symmetric? Why?

(2) [10 pts] Find the straight line $y = \alpha + \beta t$ that best fits the following data in the least squares sense:

t_i	0	1	3	6
y_i	2	3	7	12

(3) [12 pts] A student has the habit that if she doesn't study one night, she is 70% certain of studying the next night. Furthermore, the probability that she studies two nights in a row is 50%. How often does she study in the long run?

(4) [12 pts] Let

$$\mathcal{X} = \text{Span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}, \qquad \mathcal{Y} = \text{Span}\left\{\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\1\end{pmatrix}\right\}.$$

Show that $\mathcal{Y}$ is the orthogonal complement of $\mathcal{X}$ in $\mathcal{R}^4$ and calculate the matrix of the projector onto $\mathcal{X}$ along $\mathcal{Y}$.

(5) [20 pts] Let $\mathbf{A}$ be a 3×3 real-symmetric matrix with spectrum $\sigma(A) = \{-2, -4\}$ and

$$\mathcal{N}(\mathbf{A} + 2\mathbf{I}) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \right\}, \qquad \mathcal{N}(\mathbf{A} + 4\mathbf{I}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -8 \\ 6 \end{pmatrix} \right\}.$$

(a) Find a diagonal matrix D and an orthogonal matrix P so that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$.

(b) Calculate the matrices of the spectral projectors $\mathbf{G}_1$ and $\mathbf{G}_2$ corresponding to the eignevalues $\lambda_1 = -2$ and $\lambda_2 = -4$ of $\mathbf{A}$.

(c) Calculate the matrix $e^{\mathbf{A}t}$, where $t \in \mathcal{R}$.

(d) Use your answer to (c) to solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \qquad \text{where } \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(6) [12 pts]
(a) Let $\mathcal{M}$ be a vector subspace of a finite dimensional vector space $\mathcal{V}$. Define the orthogonal complement $\mathcal{M}^{\perp}$ of $\mathcal{M}$. Prove that $\mathcal{M}^{\perp}$ is a vector subspace of $\mathcal{V}$.

(b) Suppose that $\mathcal{B} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is an orthonormal basis for a vector space $\mathcal{V}$. Derive a formula for the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of a vector $\mathbf{v} \in \mathcal{V}$.

(7) [10 pts] Let $\mathbf{T} : \mathcal{U} \to \mathcal{V}$ be a linear transformation between finite-dimensional vector spaces. Let $\mathcal{B}$ be a basis for $\mathcal{U}$ and let $\mathcal{B}'$ be a basis for $\mathcal{V}$. Define the coordinate matrix $[\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$ of $\mathbf{T}$ with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$. Also, prove that for any vector $\mathbf{u} \in \mathcal{U}$,

$$[\mathbf{T}(\mathbf{u})]_{\mathcal{B}'} \;=\; [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}[\mathbf{u}]_{\mathcal{B}}.$$

(8) [12 pts] Define the concepts *algebraic multiplicity* and *geometric multiplicity*. Prove that if a square matrix is diagonalizable then for each eigenvalue the algebraic and geometric multiplicities are equal.

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________