NAME: SOLUTIONS

1	~~/24~~ 32	2	~~/16~~ 10	3	~~/12~~ 10	4	/16	5	/16	6	/16	T	/100

MATH 430 (Fall 2006) Exam II, November 1st

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [24 pts]
(a) State the three properties that characterize the determinant as a function from the space of $n \times n$ matrices to the scalars.

(I) The determinant is linear in the first row, ie

$$\det\left(\begin{bmatrix} \alpha \vec{v} + \vec{w} \\ B \end{bmatrix}\right) = \alpha \det\left(\begin{bmatrix} \vec{v} \\ B \end{bmatrix}\right) + \det\left(\begin{bmatrix} \vec{w} \\ B \end{bmatrix}\right)$$

(II) The determinant changes sign if two rows are interchanged

(III) $\det(I_n) = 1$

(b) Using (a) show that if an $n \times n$ matrix **B** is obtained from **A** by the row operation

$$\text{Row } 1 \;=\; \text{Row } 1 \;-\; \alpha \, \text{Row } 2,$$

then $\det(\mathbf{B}) = \det(\mathbf{A})$.

$$A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \qquad B = \begin{bmatrix} \vec{v}_1 - \alpha \vec{v}_2 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}$$

$$\det(B) = \det\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} - \alpha \det\begin{bmatrix} \vec{v}_2 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} = \det(A) \quad \text{by (I)}$$

0 ..

as $\det\begin{bmatrix} \vec{v}_2 \\ \vec{v}_2 \\ * \end{bmatrix} \overset{(II)}{=} -\det\begin{bmatrix} \vec{v}_2 \\ \vec{v}_2 \\ * \end{bmatrix}$ (Swap Rows 1+2) implies $\det\begin{bmatrix} \vec{v}_2 \\ \vec{v}_2 \\ * \end{bmatrix}$

(a) ~~(c)~~ Prove that similar matrices have the same spectrum.

Suppose $B = PCP^{-1}$

$\sigma(B) = \{\lambda \in \mathbb{C} \mid \det(B - \lambda I) = 0\}$

Let $\lambda \in \sigma(B)$. $\Leftrightarrow$ $\det(B - \lambda I) = 0$

$\Leftrightarrow \det(PCP^{-1} - \lambda I) = 0$

$\Leftrightarrow \det(PCP^{-1} - \lambda PP^{-1}) = 0$

$\Leftrightarrow \det[P(C - \lambda I)P^{-1}] = 0$

$\Leftrightarrow \det(P)\det(C - \lambda I)\det(P^{-1}) = 0$

$\Leftrightarrow \det(C - \lambda I) = 0$ (as $\det P \neq 0$ as P is invertible)

$\Leftrightarrow \lambda \in \sigma(C)$

(b) ~~(d)~~ Use the result of (c) to define the spectrum of a linear transformation $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ and prove that it is well defined. (Here $\mathcal{V}$ is a finite dimensional vector space.)

DEF Let B be any basis for V and let $[T]_B$ be the matrix of T in this basis. Define

$$\sigma(T) = \sigma([T]_B)$$

To show well defined ~~show~~ we know that if B' is another basis for V then $[T]_{B'} = P[T]_B P^{-1}$ for some invertible P. So ~~by (a)~~ $[T]_{B'}$ and $[T]_B$ are similar.

Therefore by (a) $\sigma([T]_{B'}) = \sigma([T]_B)$

So $\sigma(T)$ is well defined independent of choice of basis B.

$[*]$ $[*]$

2 (b) Let $\mathcal{V} = \mathcal{R}^2$,

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\}$$

and

$$\mathcal{B}' = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}.$$

Calculate the matrix $[\mathbf{T}]_{\mathcal{B}\mathcal{B}}$ of the change of basis linear transformation $\mathbf{T}$.

$$[T]_{\mathcal{BB}} = \left([\vec{x}_1]_{\mathcal{B}'} \ \ [\vec{x}_2]_{\mathcal{B}'} \right) = \left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right] \text{ where}$$

$$\left\{ \begin{aligned} \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= a \begin{pmatrix} 0 \\ 2 \end{pmatrix} + b \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 5 \end{pmatrix} &= c \begin{pmatrix} 0 \\ 2 \end{pmatrix} + d \begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{aligned} \right\} \Longrightarrow \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

So
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & -1/2 \\ 1 & 2 \end{pmatrix}$$

(b) Suppose that $\mathbf{v} \in \mathcal{V}$ has coordinate vector $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$. What is $[\mathbf{v}]_{\mathcal{B}'}$?

$$[\vec{v}]_{\mathcal{B}'} = [T]_{\mathcal{BB}} [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} -1/2 & -1/2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} -9/2 \\ 16 \end{pmatrix}$$

(3) [12 pts] Let **A** be the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 8 \end{pmatrix}.$$

Calculate det(**A**) using

(a) Row operations

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 8 \end{vmatrix} \qquad R2 \Leftrightarrow R1 \; \text{II}$$

$$= -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{vmatrix} \qquad R3 \to R3 - 4R1$$

$$= -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} \qquad R3 \to R3 + 3R2$$

$$= -2$$

(b) A cofactor expansion ALONG ROW 1

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 8 \end{vmatrix} = 0. - 1\begin{vmatrix} 1 & 3 \\ 4 & 8 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= -1(8-12) + 2(5-8)$$

$$= 4 - 6 = -2 \quad \checkmark$$

(4) [16 pts] Use eigenvalues and eigenvectors to solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$
$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$

EVALUES

$$0 = |A - \lambda I| = \begin{vmatrix} -1-\lambda & -2 \\ -2 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - 4 = (1+\lambda-2)(1+\lambda+2)$$

$$= (\lambda-1)(\lambda+3)$$

$\boxed{\lambda_1 = 1}$ $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\lambda_2 = -3}$ $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $\vec{x}(t) = c_1 e^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\boxed{\vec{x}(t) = \frac{3}{2} e^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{2} e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

[*] [*]

(5) [16 pts]
(a) Use the formula for the determinant of a block matrix to prove that if $\mathbf{B}$ is $m \times n$ and $\mathbf{C}$ is $n \times m$ then

$$\lambda^m \det(\lambda\mathbf{I}_n - \mathbf{CB}) = \lambda^n \det(\lambda\mathbf{I}_m - \mathbf{BC})$$

for all scalars λ.

If $\lambda = 0$, both sides are 0 and the equation is true.

if $\lambda \neq 0$,

$$\det\begin{pmatrix} \lambda I_m & B \\ C & I_n \end{pmatrix} = \det(\lambda I_m)\det(I_n - C(\lambda I_m)^{-1}B)$$

$\underline{\text{as } \lambda \neq 0}$

$$= \lambda^m \det(I_n - C\tfrac{1}{\lambda}I_m^{-1}B)$$

$$= \lambda^m \det\left(\left(\tfrac{1}{\lambda}I_n\right)(\lambda I_n - CB)\right)$$

$$= \lambda^m \det\left(\tfrac{1}{\lambda}I_n\right)\det(\lambda I_n - CB)$$

$$= \lambda^{m-n}\det(\lambda I_n - CB) \quad (1)$$

and

$$\det\begin{pmatrix} \lambda I_m & B \\ C & I_n \end{pmatrix} = \det(I_n)\det(\lambda I_m - BI_n^{-1}C)$$

$$= \det(\lambda I_m - BC) \quad (2)$$

So by (1) and (2)

$$\lambda^n \det(\lambda I_m - BC) = \lambda^m \det(\lambda I_n - CB)$$

(b) Use the result of (a) to show that if $n = m$ then **BC** and **CB** have the same spectrum, $\sigma(\mathbf{BC}) = \sigma(\mathbf{CB})$.

① Suppose $\lambda \in \sigma(BC)$ and $\lambda \neq 0$

Then $\det(BC - \lambda I) = 0$

So by (a) $\det(CB - \lambda I) = 0$

So $\lambda \in \sigma(CB)$

② Suppose $0 \in \sigma(BC)$

Then BC is singular. So

$$0 = \det(BC) = \det(B)\det(C) = \det(C)\det(B)$$
$$= \det(CB)$$

So CB is singular so $0 \in \sigma(CB)$

Together ① + ② show $\sigma(BC) \subseteq \sigma(CB)$.

Since argument is symmetric in B, C, $\sigma(BC) = \sigma(BC)$ must hold

(c) Construct a counterexample to show that $\sigma(\mathbf{BC}) \neq \sigma(\mathbf{CB})$ when $n \neq m$.

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \sigma(BC) = \{0, 1\}$$

$$CB = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \qquad \sigma(CB) = \{1\}$$

So $\sigma(BC) \neq \sigma(CB)$

[*] ... +2) implies det [*]

(6) [16 pts] Suppose that $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is a linear transformation such that $\mathbf{T}^2 = \mathbf{T}$. Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be a basis for $\mathcal{R}(\mathbf{T})$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_{n-r}\}$ be a basis for $\mathcal{N}(\mathbf{T})$, where $n = \dim \mathcal{V}$.

(a) Show that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r, \mathbf{y}_1, \cdots, \mathbf{y}_{n-r}\}$ are linearly independent and hence form a basis $\mathcal{B}$ for $\mathcal{V}$. [Hint: Show $\mathbf{T}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{R}(\mathbf{T})$.]

- If $\vec{x} \in R(T)$, $\vec{x} = T\vec{u}$ for some $\vec{u} \in V$

So $T\vec{x} = T(T\vec{u}) = T^2\vec{u} = T\vec{u} = \vec{x}$.

- Suppose

$$\alpha_1 \vec{x}_1 + \cdots + \alpha_r \vec{x}_r + \beta_1 \vec{y}_1 + \cdots + \beta_{n-r} \vec{y}_{n-r} = \vec{0} \quad (*)$$

Then taking T of both sides ~~and using linearity~~ ! and using linearity

$$\cancel{\boxed{\alpha_1 T(\vec{x}_1) + \cdots + \alpha_r}}$$

$$T(\alpha_1 \vec{x}_1 + \cdots + \alpha_r \vec{x}_r) + T(\beta_1 \vec{y}_1 + \cdots + \beta_{n-r} \vec{y}_{n-r}) = T(\vec{0}) = \vec{0}$$

$$\alpha_1 \vec{x}_1 + \cdots + \alpha_r \vec{x}_r + \vec{0} = \vec{0}$$

as $\vec{y}_1 \cdots \vec{y}_{n-r}$ span $N(T)$ and by Hint applied to $\vec{x} = \sum_{i=1}^{r} \alpha_i \vec{x}_i$

~~Since~~ Now since $\vec{x}_1 - \vec{x}_r$ are a basis for $R(T)$ they are LI so $\alpha_1 = \text{——} = \alpha_r = 0$ is forced.

So by $(*)$ $\beta_1 \vec{y}_1 + \cdots + \beta_{n-r} \vec{y}_{n-r} = \vec{0}$.

Finally as $\vec{y}_1 - \vec{y}_{n-r}$ is a basis for $N(T)$

$\beta_1 = \cdots = \beta_{n-r} = 0$, as req'd.

(b) Show that $[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{pmatrix}$.

$$[T]_{\mathcal{B}} = \left([T(\vec{x}_1)]_{\mathcal{B}} \cdots [T(\vec{x}_r)]_{\mathcal{B}} \; [T(\vec{y}_1)]_{\mathcal{B}} \cdots [T(\vec{y}_{n-r})]_{\mathcal{B}} \right)$$

$$= \left([\vec{x}_1]_{\mathcal{B}} - [\vec{x}_r]_{\mathcal{B}} , \; [\vec{0}]_{\mathcal{B}} - [\vec{0}]_{\mathcal{B}} \right)$$

$$= \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)$$

(c) Use (1d) to calculate the spectrum of the linear transformation $\mathbf{T}$.

$\sigma(T) = \sigma([T]_{\mathcal{B}})$ by (1d).

And $\det\left(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} - \lambda I_n \right) = \det \left(\begin{array}{c|c} I_r(1-\lambda) & 0 \\ \hline 0 & -\lambda I_{n-r} \end{array} \right)$

$= \det((1-\lambda) I_r) \det(-\lambda I_{n-r})$

$= (1-\lambda)^r (-\lambda)^{n-r}$

So $\sigma(T) = \{0, 1\}$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________