NAME:

1 /32	2 /10	3 /10	4 /16	5 /16	6 /16	T /100

MATH 430 (Fall 2006) Exam II, November 1st

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [32 pts]
(a) Prove that similar matrices have the same spectrum.

(b) Use the result of (a) to define the spectrum of a linear transformation $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ and prove that it is well defined. (Here $\mathcal{V}$ is a finite dimensional vector space.)

(c) State the three properties that characterize the determinant as a function from the space of $n \times n$ matrices to the scalars.

(d) Using (c) show that if an $n \times n$ matrix $\mathbf{B}$ is obtained from $\mathbf{A}$ by the row operation

$$\text{Row } 1 \;=\; \text{Row } 1 \;-\; \alpha \, \text{Row } 2,$$

then $\det(\mathbf{B}) = \det(\mathbf{A})$.

(2) [10 pts]
(a) Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\}$$

and

$$\mathcal{B}' = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

be two bases for $\mathcal{V} = \mathcal{R}^2$. Calculate the matrix $[\mathbf{T}]_{\mathcal{B}\mathcal{B}}$ of the change of basis linear transformation $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ from $\mathcal{B}$ to $\mathcal{B}'$.

(b) Suppose that $\mathbf{v} \in \mathcal{V}$ has coordinate vector $[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$. What is $[\mathbf{v}]_{\mathcal{B}'}$?

(3) [10 pts] Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 8 \end{pmatrix}.$$

Calculate $\det(\mathbf{A})$ using

(a) Row operations

(b) A cofactor expansion

(4) [16 pts] Use eigenvalues and eigenvectors to solve the initial value problem

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{A}\mathbf{x} \\ \mathbf{x}(0) &= \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \end{aligned}$$

where $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$.

(5) [16 pts]
(a) Use the formula for the determinant of a block matrix to prove that if $\mathbf{B}$ is $m \times n$ and $\mathbf{C}$ is $n \times m$ then

$$\lambda^m \det(\lambda \mathbf{I}_n - \mathbf{CB}) \;=\; \lambda^n \det(\lambda \mathbf{I}_m - \mathbf{BC})$$

for all scalars λ.

(b) Use the result of (a) to show that if $n = m$ then $\mathbf{BC}$ and $\mathbf{CB}$ have the same spectrum, $\sigma(\mathbf{BC}) = \sigma(\mathbf{CB})$.

(c) Construct a counterexample to show that $\sigma(\mathbf{BC}) \neq \sigma(\mathbf{CB})$ when $n \neq m$.

(6) [16 pts] Suppose that $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is a linear transformation such that $\mathbf{T}^2 = \mathbf{T}$.
Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be a basis for $\mathcal{R}(\mathbf{T})$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_{n-r}\}$ be a basis for $\mathcal{N}(\mathbf{T})$, where $n = \dim \mathcal{V}$.

(a) Show that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r, \mathbf{y}_1, \cdots, \mathbf{y}_{n-r}\}$ are linearly independent and hence form a basis $\mathcal{B}$ for $\mathcal{V}$.
[Hint: Show $\mathbf{T}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{R}(\mathbf{T})$.]

(b) Show that $[\mathbf{T}]_{\mathcal{B}} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

(c) Use (1b) to calculate the spectrum of the linear transformation $\mathbf{T}$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________