NAME:

1 /30	2 /15	3 /15	4 /10	5 /10	6 /10	7 /10	T /100

MATH 430 (Fall 2006) Exam 1, October 4th

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [30 pts]
(a) Define the term *maximal linearly independent set.*

A subset $B = \{\vec{b}_1, \dots \vec{b}_n\}$ ~~of~~ a FDVS V is a MLI set if
① B is a LI set
② If $B' = \{\vec{a}_1, \dots \vec{a}_m\}$ is ANY other LI set ~~for~~ of V then $m \leq n$.

(b) State the Basis Characterization Theorem.

Let V be a F.D.V.S, and let $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\} \subseteq V$

TFAE
① B is a basis for V
② B is a maximal LI set for V
③ B is a minimal spanning set for V

(c) State the definition of a least squares solution of a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

A vector $\vec{x}$ is a least squares solution of $A\vec{x} = \vec{b}$ if it minimizes the function

$$\mathcal{E}(\vec{x}) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (A\vec{x} - \vec{b})_i^2 = (A\vec{x} - \vec{b})^T (A\vec{x} - \vec{b})$$

(d) Suppose that $\mathbf{B}_{r\times r}$ is an invertible $r \times r$ matrix and that $\mathbf{0}_{p\times q}$ is the $p \times q$ zero matrix. Let $\mathbf{A}$ be the square matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{B}_{r\times r} & \mathbf{0}_{r\times s} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times s} \end{pmatrix}.$$

Find bases for the nullspace, $N(\mathbf{A})$, and the range, $R(\mathbf{A})$, of $\mathbf{A}$ and verify that the Rank and Nullity Theorem holds for $\mathbf{A}$. $\vec{x} \in \mathbb{R}^r$ $\vec{y} \in \mathbb{R}^s$

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} B\vec{x} \\ \vec{0} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} \quad \text{iff} \quad \vec{x} \in N(B) = \{\vec{0}\}$$

as B is invertible

So $\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in N(A)$ iff $\vec{x} = \vec{0}$.

So basis for $N(A)$ is $\{\vec{e}_{r+1}, \ldots, \vec{e}_{r+s}\}$

$$\boxed{\begin{array}{l} \text{R+N THM} \\ \dim N(A) = s \\ \dim R(A) = r \\ n = s + r \end{array}}$$

$\begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \in R(A)$ means $\begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} B\vec{x} \\ \vec{0} \end{pmatrix}$

So $\vec{v} = 0$ and $\vec{u} = B\vec{x}$ must hold, ie $\vec{u} \in R(B)$

So if $B = [B_{*1} \ldots B_{*r}]$ decomposes B into columns

Then the vectors $B_{*1} \ldots B_{*r}$ are LI as B is invertible. So

$\left\{ \begin{pmatrix} B_{*1} \\ 0_{s\times 1} \end{pmatrix} \ldots \begin{pmatrix} B_{*r} \\ 0_{s\times 1} \end{pmatrix} \right\}$ are a basis for $R(A)$

(1e) If $N(A) = N(B)$ for two matrices does $\text{Rank}(A) = \text{Rank}(B)$?

<u>Yes</u> Let A be $m \times n$, B $k \times l$.

Then $N(A) \subseteq \mathbb{R}^n$ $\quad N(B) \subseteq \mathbb{R}^l$

Since $N(A) = N(B)$ they must both contain vectors with same # of compnents

So $\quad n = l$ must hold.

So RTN THM gives

$$\dim N(A) + \text{Rank}(A) = n$$
$$\dim N(B) + \text{Rank}(B) = n$$

So $\quad \text{Rank}(A) = \text{Rank}(B)$

(1f). We did This in class.

(2) [15 pts] Let **A** be the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Find bases for the nullspace and range of the **A** and for the range of $\mathbf{A}^{\mathbf{T}}$.

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \sim \quad \begin{matrix} R2 \to R2 - 2R1 \\ R3 \to R3 - 3R1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad \begin{matrix} R2 \to -\frac{1}{3} R2 \\ R3 \to R3 + 5R2 \end{matrix}$$

$$\sim \begin{pmatrix} \boxed{1} & 2 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U \quad R1 \to R1 - 2R2$$

$R(A)$ has basis given by pivot columns in A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

x_1, x_3 basic variables, x_2, x_4 free variables.

Solution to $A\vec{x} = \vec{0}$ is

$$x_3 = -x_4$$
$$x_1 = -2x_2 - 3x_4$$

So $\vec{x} = \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Basis for $N(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Basis for $R(A^T)$ = Non zero rows of U

$= \{ (1, 2, 0, 3)$ and $(0, 0, 1, 1) \}$

$$38-\frac{9}{25}$$
$$=37\tfrac{16}{25}$$

$$13-\frac{21}{25}$$
$$=12\,4/25$$

(3) [15 pts] Find the least squares solutions to the linear system

$$\begin{aligned} 2x+3y&=2\\ 4x-2y&=-1\\ x+5y&=1\\ 2x+0y&=3 \end{aligned}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} \quad \text{ie} \quad A\vec{x}=\vec{b}.$$

Least squares solutions of $A\vec{x}=\vec{b}$ are precisely solutions of Normal equations $A^TA\vec{x} = A^T\vec{b}$:

$$A^TA = \begin{pmatrix} 2 & 4 & 1 & 2 \\ 3 & -2 & 5 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 3 \\ 3 & 38 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 2 & 4 & 1 & 2 \\ 3 & -2 & 5 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 25 & 3 & 7 \\ 3 & 38 & 13 \end{array}\right) \quad R2 \to R2 - \frac{3}{25}R1$$

$$\sim \left(\begin{array}{cc|c} 25 & 3 & 7 \\ 0 & 37\frac{16}{25} & 12\,4/25 \end{array}\right)$$

$$37\tfrac{16}{25}\,y = 12\,4/25$$

$$y = \frac{12\times25+4}{25}\cdot\frac{25}{37.25+16}$$

$$y = \frac{12\times 25+4}{37\times25+16}$$

$$25x = 7-3y$$

$$x = \frac{7-3\left(\frac{12\times25+4}{37\times25+16}\right)}{25}$$

(4) [10 pts] Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices with the same number of columns and let $\mathbf{C} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$. Prove that $N(\mathbf{C}) \subseteq N(\mathbf{A})$. Is $N(\mathbf{C}) = N(\mathbf{A})$? Why?

~~Let $\vec{x} = \begin{pmatrix} y \\ z \end{pmatrix} \in N(C)$~~

Let $\vec{x} \in N(C)$.

SO $\vec{0} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} = C\vec{x} = \begin{pmatrix} A \\ B \end{pmatrix} \vec{x} = \begin{pmatrix} A\vec{x} \\ B\vec{x} \end{pmatrix}$

So $A\vec{x} = \vec{0}$ must hold

Therefore $\vec{x} \in N(A)$

So $N(C) \subset N(A)$.

$N(C) \neq N(A)$: as can be seen by the following example

$A = [0]$ (1×1) , $B = [1]$ (1×1) , $C = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 2×1

~~$N(A) = \{\vec{0}\}$~~ $N(A) = \mathbb{R}$

$N(C) = \{ x \in \mathbb{R} / \begin{pmatrix} 1 \\ 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$ ~~$\$$~~ $= \{ x \in \mathbb{R} / \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

~~$= \{\,\}$~~ $= \{0\}$. $\boxed{\{0\} \neq \mathbb{R}}$

(5) [10 pts] Suppose that $\mathbf{A}$ is a 2×2 matrix with

$$\mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} \qquad \mathbf{A}\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}6\\7\end{pmatrix}.$$

Without working out the entries of $\mathbf{A}$, find $\mathbf{A}\begin{pmatrix}6\\10\end{pmatrix}$.

$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

Let's write $\begin{pmatrix}6\\10\end{pmatrix} = x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Solving for $\begin{pmatrix}x\\y\end{pmatrix}$:

$$\left(\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 5 & 10 \end{array}\right) \quad R2 \to R2 - 2R1$$

$$\sim \left(\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -2 \end{array}\right) \quad R1 \to R1 - 2R2$$

gives $y = -2$, $x = 10$

So

$$\begin{pmatrix}6\\10\end{pmatrix} = 10\begin{pmatrix}1\\2\end{pmatrix} + -2\begin{pmatrix}2\\5\end{pmatrix}$$

So by linearity of matrix multiplication.

$$A\begin{pmatrix}6\\10\end{pmatrix} = 10\,A\begin{pmatrix}1\\2\end{pmatrix} + -2\,A\begin{pmatrix}2\\5\end{pmatrix}$$

$$= 10\begin{pmatrix}2\\3\end{pmatrix} - 2\begin{pmatrix}6\\7\end{pmatrix}$$

$$= \begin{pmatrix}8\\16\end{pmatrix}$$

(6) [10 pts] Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for $\mathcal{R}^n$ such that $\{\mathbf{v}_{r+1}, \cdots, \mathbf{v}_n\}$ is a basis for $N(\mathbf{A})$. Prove that $\{\mathbf{A}\mathbf{v}_1, \cdots, \mathbf{A}\mathbf{v}_r\}$ is a basis for $R(\mathbf{A})$.

SPANNING

Let $\vec{w} \in R(A)$.

So $\vec{w} = A\vec{u}$ for some $\vec{u} \in \mathbb{R}^n$.

Since $\{\vec{v}_1 - \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ $\exists \alpha_1 \cdots \alpha_n \in \mathbb{R}$:

$$\vec{u} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$$

Then
$$A\vec{u} = \alpha_1 A\vec{v}_1 + \cdots + \alpha_r A\vec{v}_r + \alpha_{r+1} A\vec{v}_{r+1} + \cdots + \alpha_n A\vec{v}_n$$
$$= \alpha_1 A\vec{v}_1 + \cdots + \alpha_r A\vec{v}_r + \vec{0} + \cdots + \vec{0}$$

as $A\vec{v}_j = \vec{0}$ as $\vec{v}_j \in N(A)$ for $j > r$.

So $\vec{w} = A\vec{u} \in \mathrm{Span}\{A\vec{v}_1 \cdots A\vec{v}_r\}$

LI

Suppose $\alpha_1 A\vec{v}_1 + \cdots + \alpha_r A\vec{v}_r = \vec{0}$

Then $A(\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r) = \vec{0}$

So $\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r \in N(A)$

Since $\vec{v}_{r+1}, \cdots, \vec{v}_n$ is a basis for $N(A)$ $\exists \beta_{r+1} \cdots \beta_n$:

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r = \beta_{r+1} \vec{v}_{r+1} + \cdots + \beta_n \vec{v}_n$$

So $\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r - \beta_{r+1} \vec{v}_{r+1} - \cdots - \beta_n \vec{v}_n = \vec{0}$

So $\alpha_1 = \cdots = \alpha_r = \beta_{r+1} = \cdots = \beta_n = 0$ as $\vec{v}_1 \cdots \vec{v}_n$ BASIS

So $\alpha_1 = \cdots = \alpha_r = 0$

So $\{A\vec{v}_1 \cdots A\vec{v}_r\}$ are LI

(7) [10 pts] Let $\mathbf{v}$, $\mathbf{w}$ be two column vectors in $\mathcal{R}^n$ and let I denote the $n \times n$ identity matrix. Suppose that $\mathbf{w}^T\mathbf{v} \neq 1$. Show that the matrix $I - \mathbf{v}\mathbf{w}^T$ is invertible and that its inverse is a matrix of the form $I - c\mathbf{v}\mathbf{w}^T$, for some scalar c. Also, find a formula for c in terms of $\mathbf{v}$ and $\mathbf{w}$.

$I - \vec{v}\vec{w}^T$ is invertible if $\exists B$:

$$(I - \vec{v}\vec{w}^T)B = I = B(I - \vec{v}\vec{w}^T)$$

Lets try $B = I - c\vec{v}\vec{w}^T$.

Well

$$(I - \vec{v}\vec{w}^T)(I - c\vec{v}\vec{w}^T) = I - \vec{v}\vec{w}^T - c\vec{v}\vec{w}^T + c\,\vec{v}\underbrace{\vec{w}^T\vec{v}}_{1\times 1}\vec{w}^T$$

$$= I - (1+c)\vec{v}\vec{w}^T + (c\,\vec{w}^T\vec{v})\,\vec{v}\vec{w}^T$$

$$= I + (-1 - c + c\vec{w}^T\vec{v})\,\vec{v}\vec{w}^T = I$$

(*)

iff $-1 - c + c\vec{w}^T\vec{v} = 0$

$\Leftrightarrow \quad 1 = -c(1 - \vec{w}^T\vec{v})$

$\Leftrightarrow \quad c = \dfrac{1}{\vec{w}^T\vec{v} - 1}$ provided $\vec{w}^T\vec{v} \neq 1$ as we assumed.

So $(I - \vec{v}\vec{w}^T)^{-1} = I - \dfrac{\vec{v}\vec{w}^T}{\vec{w}^T\vec{v} - 1}$ should hold.

Indeed if you redo calculation (*) with $c = \frac{1}{\vec{w}^T\vec{v} - 1}$

Pledge: *I have neither given nor received aid on this exam*

you see that you get I, as required.

Signature: ______________________

Similarly (PTO)

$$(I - c\vec{v}\vec{w}^T)(I - \vec{v}\vec{w}^T)$$

$$= I - c\vec{v}\vec{w}^T - \vec{v}\vec{w}^T + c\vec{v}\underbrace{\vec{w}^T\vec{v}}_{1\times 1}\vec{w}^T$$

$$= I + (-c - 1 + c\vec{w}^T\vec{v})\vec{v}\vec{w}^T$$

$$= I \quad \text{if} \quad c = \frac{1}{\vec{w}^T\vec{v} - 1}$$ as required.

Together these results show~~s~~ that

$$\cancel{I - \vec{v}\vec{w}} \quad (I - \vec{v}\vec{w}^T)^{-1} = I - \frac{\vec{v}\vec{w}^T}{\vec{w}^T\vec{v} - 1}.$$