NAME:

1 /30	2 /15	3 /15	4 /10	5 /10	6 /10	7 /10	T /100

MATH 430 (Fall 2006) Exam 1, October 4th

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [30 pts]
(a) Define the term *maximal linearly independent set*.

(b) State the Basis Characterization Theorem.

(c) State the definition of a least squares solution of a linear system $\mathbf{Ax} = \mathbf{b}$.

(d) Suppose that $\mathbf{B}_{r\times r}$ is an invertible $r \times r$ matrix and that $\mathbf{0}_{p\times q}$ is the $p \times q$ zero matrix. Let $\mathbf{A}$ be the square matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{B}_{r\times r} & \mathbf{0}_{r\times s} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times s} \end{pmatrix}.$$

Find bases for the nullspace, $N(\mathbf{A})$, and the range, $R(\mathbf{A})$, of $\mathbf{A}$ and verify that the Rank and Nullity Theorem holds for $\mathbf{A}$.

(e) Suppose that $N(\mathbf{A}) = N(\mathbf{B})$ for two matrices $\mathbf{A}$ and $\mathbf{B}$. Must it be true that $\text{Rank}(\mathbf{A}) = \text{Rank}(\mathbf{B})$?

(f) Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for a vector space $\mathcal{V}$ and let $\mathbf{u} \in \mathcal{V}$. Prove that there are unique scalars $\alpha_1, \cdots, \alpha_n \in \mathcal{R}$ so that

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_1 \mathbf{v}_n.$$

(2) [15 pts] Let **A** be the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Find bases for the nullspace and range of the **A** and for the range of $\mathbf{A}^\mathbf{T}$.

(3) [15 pts] Find the least squares solutions to the linear system

$$
\begin{aligned}
2x + 3y &= 2 \\
4x - 2y &= -1 \\
x + 5y &= 1 \\
2x + 0y &= 3
\end{aligned}
$$

(4) [10 pts] Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices with the same number of columns and let $\mathbf{C} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$. Prove that $N(\mathbf{C}) \subseteq N(\mathbf{A})$. Is $N(\mathbf{C}) = N(\mathbf{A})$? Why?

(5) [10 pts] Suppose that $\mathbf{A}$ is a 2×2 matrix with

$$\mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} \qquad \mathbf{A}\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}6\\7\end{pmatrix}.$$

Without working out the entries of $\mathbf{A}$, find $\mathbf{A}\begin{pmatrix}6\\10\end{pmatrix}$.

(6) [10 pts] Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for $\mathcal{R}^n$ such that $\{\mathbf{v}_{r+1}, \cdots, \mathbf{v}_n\}$ is a basis for $N(\mathbf{A})$. Prove that $\{\mathbf{A}\mathbf{v}_1, \cdots, \mathbf{A}\mathbf{v}_r\}$ is a basis for $R(\mathbf{A})$.

(7) [10 pts] Let $\mathbf{v}$, $\mathbf{w}$ be two column vectors in $\mathcal{R}^n$ and let I denote the $n \times n$ identity matrix. Suppose that $\mathbf{w}^T\mathbf{v} \neq 1$. Show that the matrix $I - \mathbf{v}\mathbf{w}^T$ is invertible and that its inverse is a matrix of the form $I - c\mathbf{v}\mathbf{w}^T$, for some scalar c. Also, find a formula for c in terms of $\mathbf{v}$ and $\mathbf{w}$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ____________________________________