NAME: SOLUTIONS

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MATH 430 (Fall 2005) Exam 2, November 3rd

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [30 pts]
(a) Let $\mathbf{u}$ be a non-zero $n \times 1$ column vector and $\mathbf{v}$ a non-zero $m \times 1$ column vector. Prove that $\mathbf{uv}^T$ has rank 1.

PROOF 1 $\text{Range}(\vec{u}\,\vec{v}^T) = \{\vec{y} \in \mathbb{R}^n / \vec{y} = \vec{u}\,\underbrace{\vec{v}^T \vec{x}}_{1\times 1} \text{ for some } \vec{x} \in \mathbb{R}^m\}$

$= \{\vec{y} \in \mathbb{R}^n / \vec{y} = (\vec{v}^T\vec{x})\vec{u} \text{ for some } \vec{x} \in \mathbb{R}^m\}$

Since $\vec{v} \neq \vec{0}$, $\forall \alpha \in \mathbb{R}$ $\exists\, \vec{x} \in \mathbb{R}^n$ with $\alpha = \vec{v}^T\vec{x}$.

So $\text{Range}(\vec{u}\,\vec{v}^T) = \{\vec{y} \in \mathbb{R}^n / \vec{y} = \alpha\vec{u} \text{ for } \alpha \in \mathbb{R}\}$

$= \text{Span}(\vec{u})$ is 1 D as $\vec{u} \neq \vec{0}$

So $\text{Rk}(\vec{u}\,\vec{v}^T) = \dim \text{Range}(\vec{u}\,\vec{v}^T) = 1$

PROOF 2 (PTO)

(b) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. Prove that $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$.

$(AB)_{ij} = A_{i*} B_{*j}$ is ith row of A by jth col of B

$(-)\ (|)$

So $(AB)_{ii} = A_{i*} B_{*i}$

So $\text{Trace}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki}$

$= \sum_{k=1}^n \sum_{i=1}^n B_{ki} A_{ik} = \sum_{k=1}^n B_{k*} A_{*k} = \sum_{k=1}^n (BA)_{kk}$ $= \text{Trace}(BA)$

(1a) PROOF 2

$$\vec{u}\,\vec{v}^T = \begin{pmatrix} u_1 \\ | \\ u_n \end{pmatrix} (v_1 \text{ — } v_m) = \begin{pmatrix} u_1 \vec{v}^T \\ u_2 \vec{v}^T \\ \vdots \\ u_m \vec{v}^T \end{pmatrix}$$

Since $\vec{u} \neq 0$ there is a j with $u_j \neq 0$.

Since switching rows of a matrix does not change its rank we can assume $u_1 \neq 0$.

Then any other row of $\vec{u}\,\vec{v}^T$ is a multiple of the first row as

$$u_k \vec{v}^T = \left(\frac{u_k}{u_1}\right) u_1 \vec{v}^T$$

So doing row operations

Row k = Row k $-$ $\left(\frac{u_k}{u_1}\right)$ Row 1

yields the matrix $\begin{pmatrix} u_1 \vec{v}^T \\ \hline 0 \\ | \\ 0 \end{pmatrix}$

which has rank 1 since $u_1 \neq 0$ and $\vec{v} \neq 0$.

(c) Let $T : \mathcal{V} \to \mathcal{V}$, be a linear operator, where $\mathcal{V}$ is a finite dimensional vector space. Using (b), define trace(T).

Let $\mathcal{B}$ be any basis for V; and let $[T]_{\mathcal{B}}$ be the matrix of T in this basis. Define

$$\boxed{\text{trace}(T) = \text{trace}([T]_{\mathcal{B}}) \quad (*)}$$

CHECK $(*)$ is well defined, independent of choice of basis $\mathcal{B}$

If $\mathcal{B}'$ were any other basis for V then

$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P$ for some invertible P.

So $\text{Trace}([T]_{\mathcal{B}'}) = \text{Trace}(P^{-1}[T]_{\mathcal{B}}P) \overset{(b)}{=} \text{Trace}([T]_{\mathcal{B}}PP^{-1})$

$= \text{Trace}([T]_{\mathcal{B}})$. So def^n $(*)$ is indep^t of choice of basis $\mathcal{B}$.

(d) State the three properties that characterize the determinant as a function from the space of $n \times n$ real matrices to $\mathcal{R}$.

(I) det depends linearly on 1st row

ie If $A = \begin{bmatrix} \vec{u} \\ \hline M \end{bmatrix}$, $B = \begin{bmatrix} \vec{v} \\ \hline M \end{bmatrix}$, $C = \begin{bmatrix} \alpha\vec{u} + \beta\vec{v} \\ \hline M \end{bmatrix}$

Then $\det(C) = \alpha \det(A) + \beta \det(B)$

(II) det changes sign when two rows of the matrix are swapped

(III) $\det(I_{n\times n}) = 1$

(e) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices. Using the definition you gave in (d) to prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Since B is invertible, $\det B \neq 0$.

Let $d(A) = \dfrac{\det(AB)}{\det(B)}$

If we can show $d(A)$ satisfies I, II, III of (d)

Then $d(A) = \det(A)$ by uniqueness of det.

III $\det(I) = \dfrac{\det(IB)}{\det B} = 1$

II $(AB)_{i*} = A_{i*}B$. So if we swap two rows of A to get $\tilde{A}$ we must swap some two rows of AB to get $\tilde{A}B$. So ~~det~~ $d(\tilde{A}) = \dfrac{\det(\tilde{A}B)}{\det B} = \dfrac{\det(AB)}{\det B} = d(A)$

By property II applied to $\det(AB)$

PTO

(f) Let $\mathbf{u}$ be a length one vector in $\mathcal{R}^n$, and let $\mathbf{R}$ be the $n \times n$ matrix $\mathbf{R} = \mathbf{I}_n - 2\mathbf{u}\mathbf{u}^T$. Calculate $\det(\mathbf{R})$, and explain the physical meaning of the linear operator defined by $\mathbf{R}(\mathbf{v}) = \mathbf{R}\mathbf{v}$.

$R = I - 2\vec{u}\vec{u}^T$ is a rank 1 update of I.

Hence $\det(R) = 1 - 2\vec{u}^T\vec{u}$ $\qquad \|\vec{u}\| = 1$

$= 1 - 2\|\vec{u}\|^2 = 1 - 2 \cdot 1 = -1$

R is the reflection over the plane through the origin whose normal vector is $\vec{u}$.

(I) The proof of (I) is similar to that of (II).

Specifically:

Suppose $E = \begin{pmatrix} \vec{u} \\ \hline M \end{pmatrix}$, $F = \begin{pmatrix} \vec{v} \\ \hline M \end{pmatrix}$, $G = \begin{pmatrix} \alpha \vec{u} + \beta \vec{v} \\ \hline M \end{pmatrix}$

Then as $(EB)_{i*} = E_{i*} B$ we have

$$EB = \begin{pmatrix} \vec{u} B \\ \hline M \end{pmatrix}, \quad FB = \begin{pmatrix} \vec{v} B \\ \hline M \end{pmatrix}, \quad GB = \begin{pmatrix} \alpha (\vec{u} B) + \beta (\vec{v} B) \\ \hline M \end{pmatrix}$$

Since det depends linearly on 1st row (I) we have

$$\boxed{\cancel{\det(EG) = \alpha \det(EB) + \beta \det(EF)}}$$

~~So~~ $\cancel{\dfrac{\det(EG)}{\det(E)}}$

$$\det(GB) = \det(EB) + \det(FB)$$

$$\frac{\det(GB)}{\det(B)} = \frac{\det(EB)}{\det B} + \frac{\det(FB)}{\det B}$$

$$d(G) = d(E) + d(F)$$

So Prop I holds for d. ✓

(2) [10 pts] True or false? If true give a brief justification. If false provide a counterexample.

$\boxed{\text{FALSE}}$ (a) $\det(\mathbf{A}+\mathbf{B})\det(\mathbf{A}-\mathbf{B}) = \det(\mathbf{A}^2-\mathbf{B}^2)$.

NOTICE that $(A+B)(A-B) = A^2 + BA - AB - B^2 \neq A^2 - B^2$

unless $AB = BA$.

So for a counterexample we need A, B that do not commute.

EX $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\det(A+B) = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

$\det(A-B) = \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = +1$

$\det(A+B)\det(A-B) = -1$

~~det~~ But $A^2 = B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ So $\det(A^2 - B^2) = 0 \neq -1$.

(b) Let $\mathbf{v} = (2,3)^T$. In the standard basis $\mathcal{B}$ for $\mathcal{R}^2$, the matrix of the projection operator $\mathbf{P_v} : \mathcal{R}^2 \to \mathcal{R}^2$ onto the span of $\mathbf{v}$ is

$$[\mathbf{P_v}]_{\mathcal{B}} = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}.$$

The Projection operator $P_{\vec{v}}$ has the property

that $P_{\vec{v}}(\vec{v}) = \vec{v}$. $\Rightarrow [P_{\vec{v}}]_{\mathcal{B}}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

~~NOT~~ BUT $[P_{\vec{v}}]\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 26 \\ 39 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

So $\boxed{\text{FALSE}}$

In fact $[P_{\vec{v}}] = \dfrac{\vec{v}\,\vec{v}^T}{|\vec{v}|^2} = \dfrac{1}{13}\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$

(3) [12 pts] For the linear operator $\mathbf{T} : \mathcal{R}^2 \to \mathcal{R}^2$ defined by $\mathbf{T}(x, y) = (x - y, 2x + 4y)$, calculate the matrix, $[\mathbf{T}]_{\mathcal{B}}$, of $\mathbf{T}$ in the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

$[T] = [T]_B$ satisfies

$$T(\vec{u}_i) = \sum_{j=1}^{n} [T]_{ji} \, \vec{u}_j$$

where $\vec{u}_1, \vec{u}_2$ is the basis B.

So

$$T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 6 \end{pmatrix} = [T]_{11} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + [T]_{21} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} [T]_{*1}$$

$$T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 8 \end{pmatrix} = [T]_{12} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + [T]_{22} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} [T]_{*2}$$

<u>NOW</u> to find $[T]_{*1}$:

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & 6 \end{array}\right) \xrightarrow[R2 = -R2]{R2 = R2 - R1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & +1 & -6 \end{array}\right) \xrightarrow{R1 = R1 - 2R2} \left(\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & -6 \end{array}\right)$$

So $[T]_{*1} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}$

AND for $[T]_{*2}$ using same row opns

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & 8 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & -7 \end{array}\right).$$

So

$$\boxed{[T]_B = \begin{pmatrix} 12 & 15 \\ -6 & -7 \end{pmatrix}}$$

(4) [18 pts] Let $\mathbf{P}$ be the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(a) Calculate $\det(\mathbf{P})$ using

(i) Row operations

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 6 & 6 \\ 7 & 8 & 9 \end{vmatrix} \underset{=}{R3 \to R3 - 7R1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 6 & 6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$\underset{=}{R3 \to R3 + R2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 6 & 6 \\ 0 & 0 & -6 \end{vmatrix} = 1 \cdot 6 \cdot (-6) = -36$$

(ii) Block determinants based on the blocking

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \qquad \text{where } \mathbf{A} \text{ is } 1 \times 1 \text{ and } \mathbf{D} \text{ is } 2 \times 2.$$

Since A is 1×1, $A^{-1} = A = [1]$. So lets use

$$\det(P) = \det(A)\det(D - CA^{-1}B)$$

$$= 1 \cdot \det(D - CB)$$

$$= \left| \begin{pmatrix} 6 & 6 \\ 8 & 9 \end{pmatrix} - \begin{pmatrix} 0 \\ 7 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 6 & 6 \\ 8 & 9 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 14 & 21 \end{pmatrix} \right| = \begin{vmatrix} 6 & 6 \\ -6 & -12 \end{vmatrix} = -36$$

(iii) A cofactor expansion.

Since there is a 0 in Col 1 (and Row 2)

use cofactor expansion based on Col 1 (or Row 2)

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 6 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 6 & 6 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 6 & 6 \end{vmatrix}$$

$$= 54 - 48 + 7(12 - 18)$$

$$= -36$$

(b) What is $\det(\mathbf{P}^T\mathbf{P})$, and why?

$\det(P^T P) = \det(P^T)\det(P)$ by Product Rule

$= (\det(P))^2$ as $\det(P^T) = \det(P)$

(5) [10 pts] Let $\mathbf{T} : \mathcal{V} \to \mathcal{W}$ be a linear transformation between finite-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$. Let $\mathcal{B}$ be a basis for $\mathcal{V}$ and let $\mathcal{B}'$ be a basis for $\mathcal{W}$. Define the matrix $[\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$ of $\mathbf{T}$ with respect to these two bases, and prove that

$$[\mathbf{T}(\mathbf{u})]_{\mathcal{B}'} = [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}[\mathbf{u}]_{\mathcal{B}}.$$

Let $\vec{v}_1 — \vec{v}_n$ be a basis for $\mathcal{V}$

and $\vec{w}_1 — \vec{w}_m$ a basis for $\mathcal{W}$.

The matrix $[T]_{BB'}$ is defined by the equation

$$T(\vec{v}_i) = \sum_{j=1}^{m} \left([T]_{BB'}\right)_{ji} \vec{w}_j \quad ①$$

i.e. $\left([T]_{BB'}\right)_{ji}$ = jth Coefficient of $T(\vec{v}_i)$ in the basis $w_1 — w_m$.

Equivalently $[T]_{BB'} = \left([T(\vec{v}_1)]_{B'} \mid \cdots \mid [T(\vec{v}_n)]_{B'}\right)$.

Suppose $[\vec{u}]_B = \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$ so that $\vec{u} = \sum_{i=1}^{n} \alpha_i \vec{u}_i$

Then

$$T(\vec{u}) = \sum_{i=1}^{n} \alpha_i T(\vec{v}_i) \quad \text{by LINEARITY of } T$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \left([T]_{BB'}\right)_{ji} \vec{w}_j \quad \text{by ①}$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \left([T]_{BB'}\right)_{ji} \alpha_i \right) \vec{w}_j = \sum_{j=1}^{m} \left([T]_{BB'}[\vec{u}]_B\right)_j \vec{w}_j \cancel{= \sum [T(\vec{u})]_{B'}}$$

BUT

$T(\vec{u}) = \sum \left([T(\vec{u})]_{B'}\right) \times \vec{w}_j$

So by UNIQUENESS

$[T]_{BB'}[\vec{u}]_B = [T(\vec{u})]_{B'}$

(6) [8 pts] Suppose **A** is a square matrix whose entries are differentiable functions of a real variable t, that is. $\mathbf{A}_{ij} = \mathbf{A}_{ij}(t)$. Prove that $\det \mathbf{A}$ is also a differentiable function of t.

We know

$$\det(A(t)) = \sum_{p} \sigma(p)\, A_{1p_1}(t)\, A_{2p_2}(t) - A_{np_n}(t)$$

where we sum over all $n!$ permutations p of $(1, 2, \ldots, n)$ and where $\sigma(p) = \pm 1$ is the parity of p.

So $\det(A(t))$ is a polynomial in the $A_{ij}(t)$, ie $\det(A(t))$ is a sum of products of differentiable functions and so is differentiable by the sum, product and Chain Rules for differentiation.

(7) [12 pts] The least squares quadratic fit to m data points $(t_1, y_1), (t_2, y_2), \cdots (t_m, y_m)$ in $\mathcal{R}^2$ is the quadratic function $y = f(t) = \alpha + \beta t + \gamma t^2$ for which the parameter vector (α, β, γ) is the global minimum of the function

$$Q = Q(\alpha, \beta, \gamma) = \sum_{i=1}^{m} (\alpha + \beta t_i + \gamma t_i^2 - y_i)^2.$$

(a) Let $\mathbf{x} = (\alpha, \beta, \gamma)^T$. Find an $m \times 1$ vector $\mathbf{y}$ and an $m \times 3$ matrix $\mathbf{A}$ so that

$$Q = Q(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2.$$

Let $\vec{y} = \begin{pmatrix} y_1 \\ | \\ y_m \end{pmatrix}$ $A = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ | & | & | \\ 1 & t_m & t_m^2 \end{pmatrix}$

Then

$$\underset{m\times 3}{A}\ \underset{3\times 1}{\vec{x}} = \begin{pmatrix} \alpha + \beta t_1 + \gamma t_1^2 \\ \vdots \\ \alpha + \beta t_m + \gamma t_m^2 \end{pmatrix}$$

So

$$\|A\vec{x} - \vec{y}\|^2 = \sum_{i=1}^{m} \left((A\vec{x})_i - y_i\right)^2$$

$$= \sum_{i=1}^{m} \left(\alpha + \beta t_i + \gamma t_i^2 - y_i\right)^2$$

(b) By differentiating $Q(\mathbf{x})$ with respect to the i-th coordinate $\mathbf{x}_i$ of $\mathbf{x}$, prove that the minimizer of Q satifies the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{y}$.

The most elegant proof is:

$$Q(\vec{x}) = \|A\vec{x} - \vec{y}\|^2 = (A\vec{x} - \vec{y})^T (A\vec{x} - \vec{y})$$

$$= (\vec{x}^T A^T - \vec{y}^T)(A\vec{x} - \vec{y})$$

$$= \vec{x}^T A^T A \vec{x} - \vec{y}^T A \vec{x} - \vec{x}^T A \vec{y} + \vec{y}^T \vec{y}$$

$$Q(\vec{x}) = \vec{x}^T A^T A \vec{x} - 2\vec{y}^T A \vec{x} + \vec{y}^T \vec{y}$$

as $(\vec{x}^T A \vec{y}) \overset{1\times 1}{=} (\vec{x}^T A \vec{y})^T = \vec{y}^T A^T \vec{x}$

So

$$0 = \frac{\partial Q}{\partial x_i} = \frac{\partial \vec{x}^T}{\partial x_i} A^T A \vec{x} + \vec{x}^T A^T A \frac{\partial \vec{x}}{\partial x_i} - 2\vec{y}^T A \frac{\partial \vec{x}}{\partial x_i}$$

$$= e_i^T A^T A \vec{x} + \vec{x}^T A^T A \vec{e}_i - 2\vec{y}^T A \vec{e}_i$$

$$= 2\vec{e}_i^T (A^T A \vec{x} - A\vec{y})$$

$$= 2 (A^T A \vec{x} - A\vec{y})_{*i} \quad \text{for all } i = 1-m$$

So $A^T A \vec{x} = A\vec{y}$ holds

Pledge: *I have neither given nor received aid on this exam.*

Signature: ____________________