NAME:

1 /30	2 /10	3 /12	4 /18	5 /10	6 /8	7 /12	T /100

MATH 430 (Fall 2005) Exam 2, November 3rd

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [30 pts]
(a) Let $\mathbf{u}$ be a non-zero $n \times 1$ column vector and $\mathbf{v}$ a non-zero $m \times 1$ column vector. Prove that $\mathbf{u}\mathbf{v}^T$ has rank 1.

(b) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. Prove that $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$.

(c) Let $\mathbf{T} : \mathcal{V} \to \mathcal{V}$, be a linear operator, where $\mathcal{V}$ is a finite dimensional vector space. Using (b), define trace($\mathbf{T}$).

(d) State the three properties that characterize the determinant as a function from the space of $n \times n$ real matrices to $\mathcal{R}$.

(e) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices. Using the definition you gave in (d) to prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

(f) Let $\mathbf{u}$ be a length one vector in $\mathcal{R}^n$, and let $\mathbf{R}$ be the $n \times n$ matrix $\mathbf{R} = \mathbf{I}_n - 2\mathbf{u}\mathbf{u}^T$. Calculate $\det(\mathbf{R})$, and explain the physical meaning of the linear operator defined by $\mathbf{R}(\mathbf{v}) = \mathbf{R}\mathbf{v}$.

(2) [10 pts] True or false? If true give a brief justification. If false provide a counterexample.
(a) $\det(\mathbf{A}+\mathbf{B})\det(\mathbf{A}-\mathbf{B}) = \det(\mathbf{A}^2 - \mathbf{B}^2)$.

(b) Let $\mathbf{v} = (2,3)^T$. In the standard basis $\mathcal{B}$ for $\mathcal{R}^2$, the matrix of the projection operator $\mathbf{P}_\mathbf{v} : \mathcal{R}^2 \to \mathcal{R}^2$ onto the span of $\mathbf{v}$ is

$$[\mathbf{P}_\mathbf{v}]_\mathcal{B} = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}.$$

(3) [12 pts] For the linear operator $\mathbf{T} : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ defined by $\mathbf{T}(x, y) = (x - y, 2x + 4y)$, calculate the matrix, $[\mathbf{T}]_{\mathcal{B}}$, of $\mathbf{T}$ in the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(4) [18 pts] Let $\mathbf{P}$ be the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(a) Calculate $\det(\mathbf{P})$ using

(i) Row operations

(ii) Block determinants based on the blocking

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \qquad \text{where } \mathbf{A} \text{ is } 1 \times 1 \text{ and } \mathbf{D} \text{ is } 2 \times 2.$$

(iii) A cofactor expansion.

(b) What is $\det(\mathbf{P}^T\mathbf{P})$, and why?

(5) [10 pts] Let $\mathbf{T} : \mathcal{V} \to \mathcal{W}$ be a linear transformation between finite-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$. Let $\mathcal{B}$ be a basis for $\mathcal{V}$ and let $\mathcal{B}'$ be a basis for $\mathcal{W}$. Define the matrix $[\mathbf{T}]_{\mathcal{B}\mathcal{B}'}$ of $\mathbf{T}$ with respect to these two bases, and prove that

$$[\mathbf{T}(\mathbf{u})]_{\mathcal{B}'} \;=\; [\mathbf{T}]_{\mathcal{B}\mathcal{B}'}[\mathbf{u}]_{\mathcal{B}}.$$

(6) [8 pts] Suppose $\mathbf{A}$ is a square matrix whose entries are differentiable functions of a real variable t, that is, $\mathbf{A}_{ij} = \mathbf{A}_{ij}(t)$. Prove that $\det \mathbf{A}$ is also a differentiable function of t.

(7) [12 pts] The least squares quadratic fit to m data points (t_1, y_1), $(t_2, y_2), \cdots (t_m, y_m)$ in $\mathcal{R}^2$ is the quadratic function $y = f(t) = \alpha + \beta t + \gamma t^2$ for which the parameter vector (α, β, γ) is the global minimum of the function

$$Q \;=\; Q(\alpha, \beta, \gamma) \;=\; \sum_{i=1}^{m} (\alpha + \beta t_i + \gamma t_i^2 - y_i)^2.$$

(a) Let $\mathbf{x} = (\alpha, \beta, \gamma)^T$. Find an $m \times 1$ vector $\mathbf{y}$ and an $m \times 3$ matrix $\mathbf{A}$ so that

$$Q \;=\; Q(\mathbf{x}) \;=\; \|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2.$$

(b) By differentiating $Q(\mathbf{x})$ with respect to the i-th coordinate $\mathbf{x}_i$ of $\mathbf{x}$, prove that the minimizer of Q satifies the normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{y}$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________