NAME: SOLUTIONS

1 /30	2 /15	3 /15	4 /10	5 /12	6 /8	7 /10	T /100

MATH 430 (Fall 2005) Exam 1, October 6th

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [30 pts]
(a) State what it means for a set of vectors to be *linearly independent.*

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent if whenever $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0}$ it follows that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

(b) Define the term *minimal spanning set.*

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in a vector space V is a minimal spanning set for V if

① $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

and ② If $k < n$ and $\vec{w}_1, \ldots, \vec{w}_k$ are elements of V then $\vec{w}_1, \ldots, \vec{w}_k$ do not span V

(c) Suppose a 5×3 matrix $\mathbf{A}$ has rank 2. Let $\mathbf{x}_1 = (1, 0, 5)^T$ and $\mathbf{x}_2 = (0, 2, 3)^T$. Can $\mathbf{A}\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{0}$? Explain.

$A : \mathbb{R}^3 \longrightarrow \mathbb{R}^5$

So by Rank + Nullity Theorem

$\dim N(A) = 3 - Rk(A) = 3 - 2 = 1$

Now if $A\vec{x}_1 = 0$ and $A\vec{x}_2 = 0$ Then $\vec{x}_1, \vec{x}_2 \in N(A)$

But $\vec{x}_1$ and $\vec{x}_2$ are L.I since

$\alpha_1 \begin{pmatrix}1\\0\\5\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \alpha_1 = 0$ and $\alpha_2 = 0$

(d) What does it mean for a vector $\mathbf{x}$ to be a least squares solution of a linear system $\mathbf{Ax} = \mathbf{b}$?

A vector $\vec{x}$ is a least squares solⁿ of $A\vec{x} = \vec{b}$ if $\vec{x}$ minimizes the function

$$Q(\vec{x}) = (A\vec{x} - \vec{b})^T (A\vec{x} - \vec{b}).$$

(e) Suppose an $n \times n$ system $\mathbf{Ax} = \mathbf{b}$ is consistent for all vectors $\mathbf{b} \in \mathcal{R}^n$. What can you say about $N(\mathbf{A})$, and why?

$A\vec{x} = \vec{b}$ consistent $\Leftrightarrow \vec{b} \in R(A)$.

So $R(A) = \mathbb{R}^n$

So by Rank + Nullity Thm, $\dim N(A) = 0$

So $N(A) = \{0\}$

(f) The first column of $\mathbf{AB}$ is a linear combination of all the columns of $\mathbf{A}$. What are the coefficients in this combination if

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In general $(AB)_{ij} = \sum_{k=1}^{K} A_{ik} B_{kj}$

So $(AB)_{*1} = \sum_{k=1}^{K} A_{*k} B_{k1} = \sum_{k=1}^{K} B_{k1} A_{*k}$

ie Col 1 of AB is a linear combⁿ of cols of A

In our case then

$$(AB)_{*1} = B_{11} A_{*1} + B_{21} A_{*2} + B_{31} A_{*3}$$
$$= 1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

(2) [15 pts] Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix}.$$

A is $m \times n$
3×4

When Gaussian elimination is used to find a row echelon form $\mathbf{U}$ for $\mathbf{A}$, the matrix $(\mathbf{A}|\mathbf{I})$ is reduced to $(\mathbf{U}|\mathbf{P})$, where

$$\mathbf{U} = \begin{pmatrix} \boxed{1} & 1 & 2 & 1 \\ 0 & \boxed{1} & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix}.$$

Using this information, find bases for the four fundamental subspaces of $\mathbf{A}$.

BASIS FOR $R(A)$ = Basic columns in A

$= \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$ So $r = Rk(A) = 2$

BASIS for $R(A^T)$ = Non Zero Rows of U

$= \{ (1, 1, 2, 1), (0, 1, 3, -2) \}$

BASIS for $N(A^T)$ = Last $m - r = 3 - 2 = 1$ row of P

$= \{ (-3, 1, 1) \}$

BASIS for $N(A)$ = $\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

SINCE General Solution of $A\vec{x} = \vec{0}$ is

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_3 - 3x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$x_2 = -3x_3 + 2x_4$

$x_1 = -x_2 - 2x_3 - x_4 = 3x_3 - 2x_4 - 2x_3 - x_4 = x_3 - 3x_4$

(3) [15 pts] Find the least squares solutions to the linear system

$$x + 2y = 1$$
$$3x - y = 0$$
$$-x + 2y = 3$$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad \text{so} \quad A\vec{x} = \vec{b}$$

$\boxed{3 \times 2}$

Normal equations $A^T A \vec{x} = A^T \vec{b}$ are

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 11 & -3 \\ -3 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{99-9} \begin{pmatrix} 9 & 3 \\ 3 & 11 \end{pmatrix} \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

$$= \frac{1}{90} \begin{pmatrix} 6 \\ 82 \end{pmatrix} = \begin{pmatrix} \frac{1}{15} \\ \frac{41}{45} \end{pmatrix}$$

(4) [10 pts] Let $\mathbf{I}_n$ be the $n \times n$ identity matrix. Show that for any $n \times n$ matrix X that

$$\begin{pmatrix} \mathbf{X} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{I}_n & -\mathbf{X} \end{pmatrix}.$$

Does it follow that

$$\begin{pmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}?$$

$$\begin{pmatrix} X & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} 0 & I_n \\ I_n & -X \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_n & -X \end{pmatrix}\begin{pmatrix} X & I_n \\ I_n & 0 \end{pmatrix}$$

~~and~~ $\cancel{\begin{pmatrix} 0 & I_n \\ I_n & -X \end{pmatrix}\begin{pmatrix} X & I_n \\ I_n & 0 \end{pmatrix} =}$

NO $\quad m=1, \; n=2$

$$\left(\begin{array}{cc|c} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)\left(\begin{array}{cc|c} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|cc} 0 & * & * \\ \hline * & * & * \\ * & * & * \end{array}\right) \neq I_3$$

NOTE

It is not enough to observe that $I_n \, I_m$ are not conformable.

EG $\quad \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}\begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix} = I_{n+m}$

So $\begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}^{-1} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}$

although we cannot use BLOCK multiplication to establish this fact since I_n, I_m are not conformable.

5+5

- (5) [12 pts]
(a) Let $\mathbf{A}$ be the $m \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. Prove that if $N(\mathbf{A}) = \{\mathbf{0}\}$ then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent.

$$A = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n)$$

Suppose $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}$

So $\vec{0} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n) \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$

So $\begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix} \in N(A) = \{\vec{0}\}$

So $\alpha_1 = 0, \dots, \alpha_n = 0.$

So $\vec{v}_1 — \vec{v}_n$ are LI

(b) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent vectors.
Prove that if $\mathbf{w} \notin \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}$ are linearly independent.

Suppose $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n + \beta \vec{w} = \vec{0}.$

If $\beta \neq 0$ then $\vec{w} = -\frac{1}{\beta}(\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n) \in \mathrm{Span}\{\vec{v}_1 — \vec{v}_n\}$

So $\beta = 0$ is forced.

But then $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}$

So $\alpha_1 = 0 \dots, \alpha_n = 0$ as $\vec{v}_1 — \vec{v}_n$ are LI

So $\alpha_1 = \alpha_2 = \dots = \alpha_n = \beta = 0$

So $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n, \vec{w}$ are LI

(6) [8 pts] Prove that $N(\mathbf{B}) \subseteq N(\mathbf{AB})$. Is $N(\mathbf{B}) = N(\mathbf{AB})$?

Let $\vec{x} \in N(B)$

So $B\vec{x} = 0$

So $AB\vec{x} = 0$

So $\vec{x} \in N(AB)$

No Let $B = I_3$, $N(B) =$

$A = O_3$ $N(A) =$

$AB = O_3$ $N(AB) =$

(7) [10 pts]
(a) Prove that the set of symmetric matrices is a vector subspace of the vector space of all $n \times n$ matrices.

Let A_1, A_2 be Symmetric and $\alpha \in \mathbb{R}$

Then $(\alpha A_1 + A_2)^T = \alpha A_1^T + A_2^T = \alpha A_1 + A_2$

as $A_1^T = A_1$ and $A_2^T = A_2$.

So $\alpha A_1 + A_2$ is Symmetric.

So the set of symmetric matrices is a subspace.

(b) Find a basis for the vector space of all 2×2 symmetric matrices.

Let A be a symmetric 2×2 matrix.

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \overset{*}{=} a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So let $S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We have just shown these 3 matrices span the subspace of 2×2 matrices.

Suppose $\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

by $*$ $\Rightarrow$ $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

So they are LI too.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________