NAME:

1 /30	2 /15	3 /15	4 /10	5 /12	6 /8	7 /10	T /100

MATH 430 (Fall 2005) Exam 1, October 6th

Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 100 points.

(1) [30 pts]
(a) State what it means for a set of vectors to be *linearly independent*.

(b) Define the term *minimal spanning set*.

(c) Suppose a 5×3 matrix $\mathbf{A}$ has rank 2. Let $\mathbf{x}_1 = (1, 0, 5)^T$ and $\mathbf{x}_2 = (0, 2, 3)^T$.
Can $\mathbf{A}\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{0}$? Explain.

(d) What does it mean for a vector $\mathbf{x}$ to be a least squares solution of a linear system $\mathbf{Ax} = \mathbf{b}$?

(e) Suppose an $n \times n$ system $\mathbf{Ax} = \mathbf{b}$ is consistent for all vectors $\mathbf{b} \in \mathcal{R}^n$. What can you say about $N(\mathbf{A})$, and why?

(f) The first column of $\mathbf{AB}$ is a linear combination of all the columns of $\mathbf{A}$. What are the coefficients in this combination if

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(2) [15 pts] Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{pmatrix}.$$

When Gaussian elimination is used to find a row echelon form $\mathbf{U}$ for $\mathbf{A}$, the matrix $(\mathbf{A}|\mathbf{I})$ is reduced to $(\mathbf{U}|\mathbf{P})$, where

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix}.$$

Using this information, find bases for the four fundamental subspaces of $\mathbf{A}$.

(3) [15 pts] Find the least squares solutions to the linear system

$$\begin{aligned} x + 2y &= 1 \\ 3x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

(4) [10 pts] Let $\mathbf{I}_n$ be the $n \times n$ identity matrix. Show that for any $n \times n$ matrix X that

$$\begin{pmatrix} \mathbf{X} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{I}_n & -\mathbf{X} \end{pmatrix}$$

Does it follow that

$$\begin{pmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}?$$

(5) [12 pts]
(a) Let $\mathbf{A}$ be the $m \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. Prove that if $N(\mathbf{A}) = \{\mathbf{0}\}$ then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent.

(b) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent vectors.
Prove that if $\mathbf{w} \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}$ are linearly independent.

(6) [8 pts] Prove that $N(\mathbf{B}) \subseteq N(\mathbf{AB})$. Is $N(\mathbf{B}) = N(\mathbf{AB})$?

(7) [10 pts]

(a) Prove that the set of symmetric matrices is a vector subspace of the vector space of all $n \times n$ matrices.

(b) Find a basis for the vector space of all 2×2 symmetric matrices.

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________