5.9 COMPLEMENTARY SUBSPACES

Let X, Y be subspaces of a V.S. V.

PROP 1

1. $X+Y = \{\vec{x}+\vec{y} \in V \mid \vec{x} \in X, \vec{y} \in Y\}$ is a subspace of V
2. $X \cap Y = \{\vec{v} \in V \mid \vec{v} \in X \text{ and } \vec{v} \in Y\}$ is a subspace of V
3. $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$.

Pf ①, ② : $X+Y$ and $X \cap Y$ are closed under addition + scalar mult^n ✓

③ Let $S = \{\vec{z}_1 \dots \vec{z}_t\}$ be basis for $X \cap Y$.
Extend S to bases

$$B_X = \{\vec{z}_1 \dots \vec{z}_t, \vec{x}_1 \dots \vec{x}_m\} \quad \text{for } X$$

$$B_Y = \{\vec{z}_1 \dots \vec{z}_t, \vec{y}_1 \dots \vec{y}_n\} \quad \text{for } Y$$

CLAIM $\mathcal{B} = B_X \cup B_Y$ is a basis for $X+Y$.
Hence $\dim(X+Y) = t+m+n = (t+m)+(t+n)-t$ ✓

Pf. • Clearly $\mathcal{B}$ spans $\mathcal{X}+\mathcal{Y}$.

• LI

Suppose $\sum \alpha_i \vec{z}_i + \sum \beta_j \vec{x}_j + \sum \gamma_k \vec{y}_k = \vec{0}$ (*)

Then $\sum \gamma_k \vec{y}_k = -\sum \alpha_i \vec{z}_i - \sum \beta_j \vec{x}_j \in \mathcal{X} \cap \mathcal{Y}$

So $\exists \delta_i$:

$$\sum \gamma_k \vec{y}_k = \sum \delta_i \vec{z}_i$$

By LI of $\mathcal{B}_{\mathcal{Y}} = \{\vec{z}_1 \ldots \vec{z}_t, \vec{y}_1 \ldots \vec{y}_n\}$

$$\gamma_k = 0 = \delta_i$$

So by (*) $\sum \beta_j \vec{x}_j = 0$

Since $\vec{x}_j$'s are LI, $\beta_j = 0$ too $\square$

~~Ex~~ $\cancel{V = \mathbb{R}^3}$

(3)

—— o ——

DEF X, Y are <u>COMPLEMENTARY SUBSPACES OF V</u> if

① $V = X + Y$

② $X \cap Y = \{\vec{0}\}$

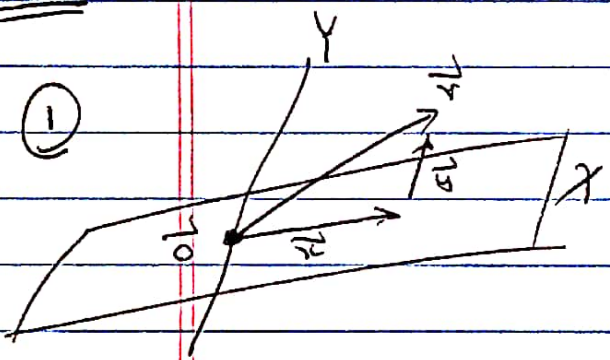
In this case we write

$$V = X \oplus Y \quad \text{A "DIRECT SUM"}$$

NOTE By PROP 1 ③ : $\boxed{\dim V = \dim X + \dim Y}$

EXS

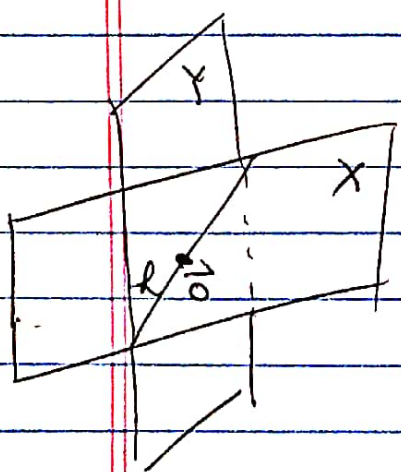
①



$V = X \oplus Y$

$\vec{v} = \vec{x} + \vec{y}$,

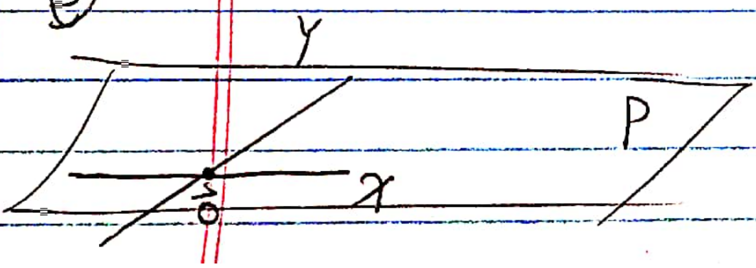
②



$X + Y = \mathbb{R}^3$

but $X \cap Y = \ell \neq \{\vec{0}\}$

③



$X + Y = P \subseteq \mathbb{R}^3$

$X \cap Y = \{\vec{0}\}$

COMPLEMENTARY SUBSPACE THM [THM 2]

Let B_X, B_Y be bases for X, Y, subspaces of V.

TFAE

1. $V = X \oplus Y$
2. $\forall \vec{v} \in V \quad \exists! \ \vec{x} \in X, \vec{y} \in Y : \vec{v} = \vec{x} + \vec{y}$
3. $B_X \cap B_Y = \phi$ and $B_X \cup B_Y$ is basis for V.

EX Let V have basis $\{\vec{v}_1, \ldots \vec{v}_n\}$

Let

$$X = \text{Span}\{\vec{v}_1, \ldots \vec{v}_k\}, \quad Y = \text{Span}\{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$$

Then

$$V = X \oplus Y$$

PF Show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

$\boxed{(1) \Rightarrow (2)}$

$\exists$ from $V = X + Y$

! If $\vec{v} = \vec{x}_1 + \vec{y}_1 = \vec{x}_2 + \vec{y}_2$

Then $\underbrace{\vec{x}_1 - \vec{x}_2}_{\in X} = \underbrace{y_2 - \vec{y}_1}_{\in Y} \in X \cap Y = \{\vec{0}\}$

So $\vec{x}_1 = \vec{x}_2$, $\vec{y}_1 = \vec{y}_2$

$\boxed{③ \Rightarrow ③}$

ⓐ $B_X \cap B_Y = \phi$.

Suppose $\exists \vec{v} \in B_X \cap B_Y \subset X \cap Y$

Then $\vec{v} \neq \vec{0}$

and $\vec{v} = \vec{v} + \vec{0} \in X + Y$

$= \vec{0} + \vec{v} \in X + Y$

So $\vec{v}$ has $\underline{\underline{2}}$ decomp^{ns} $\cancel{X}$

ⓑ $B_X \cup B_Y$ is basis for V

Write $B_X = \{\vec{x}_1, \dots \vec{x}_n\}$, $B_Y = \{\vec{y}_1, \dots, \vec{y}_m\}$

(i) SPANNING

Let $\vec{v} \in V$

By ② $\vec{v} = \vec{x} + \vec{y} = \sum \alpha_i \vec{x}_i + \sum \beta_j \vec{y}_j$

$\in \operatorname{Span}\{B_X \cup B_Y\}$

(ii) LI

IF $\sum \alpha_i \vec{x}_i + \sum \beta_j \vec{y}_j = \vec{0} = \vec{0}_X + \vec{0}_Y$

By !

$\sum \alpha_i \vec{x}_i = \vec{0} \Rightarrow \alpha_i = 0 \quad \forall i$

$\sum \beta_j \vec{y}_j = \vec{0} \Rightarrow \beta_j = 0 \quad \forall j$

$\boxed{③ \Rightarrow ①}$ SIMILAR TO PROOF OF PROP 1 ③ $\square$

DEF Suppose $V = X \oplus Y$.

The **PROJECTOR ONTO X ALONG Y** is the L.T.

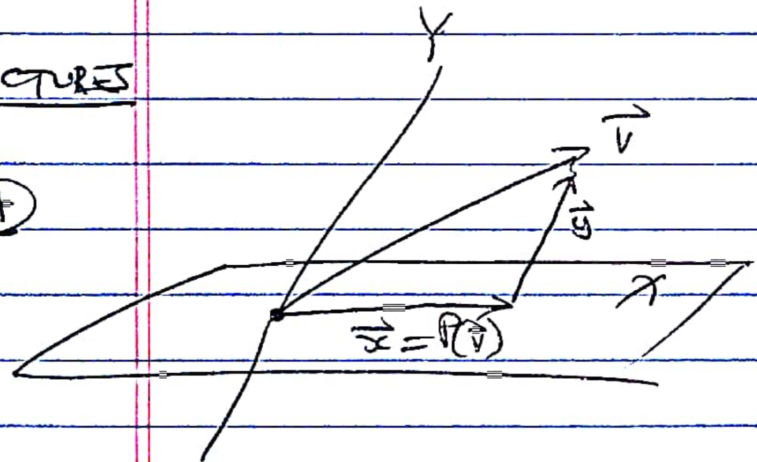
$$P : V \to V \quad \text{defined by}$$

$$P(\vec{v}) = \vec{x} \quad \text{where } \vec{v} = \vec{x} + \vec{y}$$
$$\vec{x} \in X, \ \vec{y} \in Y.$$

HWK 5.9A Verify that P is a L.T.

PICTURES

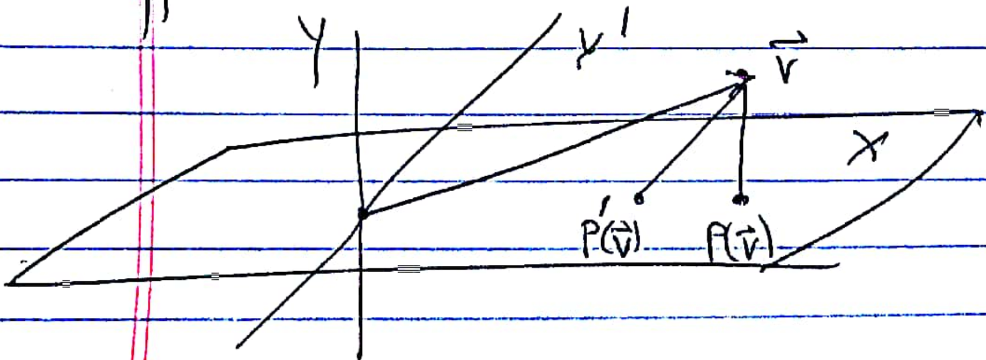
(A)



$P(\vec{v})$ PROJECTS $\vec{v}$ ONTO X ALONG LINE // TO Y.

"OBLIQUE PROJN"

(B) Suppose $\mathbb{R}^3 = X \oplus Y = X \oplus Y'$



THE PROJECTOR DEPENDS ON Y AS WELL AS X

PROP 3

Let $V = X \oplus Y$, and P PROJ^R ONTO X ALONG Y. Then

① $P^2 = P$ (P is idempotent)

② $I - P$ is projector onto Y ~~and~~ along X

③ $R(P) = N(I-P) = X$

$R(I-P) = N(P) = Y$.

PF of ③ is HWK.

PROP 4

If $P : V \to V$ has $P^2 = P$
Then P is the projector onto $R(P)$ along $N(P)$.

PF

Ⓘ CLAIM $P^2 = P \implies V = R(P) \oplus N(P)$

PF

ⓐ $V = R(P) + N(P)$ as $\vec{v} = P\vec{v} + (I-P)\vec{v}$
$\in R(P) + N(P)$

as

$P(I-P) = P - P^2 = 0$

ⓑ IF $\vec{v} \in R(P) \cap N(P)$ Then
$\vec{v} = P\vec{x} = P^2\vec{x} = P(P\vec{x}) = P\vec{v} = \vec{0}$.

II Write $\vec{v} = \vec{x} + \vec{y}$ where $\vec{x} = P\vec{u} \in R(P)$,
$\vec{y} \in N(P)$

So

$$\vec{v} = P\vec{u} + \vec{y}$$

Then $P\vec{v} = P^2\vec{u} + P\vec{y} = P^2\vec{u} = P\vec{u} = \vec{x}$ ✓

∎

THE MATRIX OF A PROJECTOR

Let $P: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Projector onto $X = R(P)$ along $Y = N(P)$.

By Complementary Subspace Thm (Thm 2)

if $\mathcal{B}_X = \{\vec{x}_1, \dots, \vec{x}_r\}$, $\mathcal{B}_Y = \{\vec{y}_1, \dots, \vec{y}_{n-r}\}$ are

bases for X, Y then $\mathcal{B} = \mathcal{B}_X \cup \mathcal{B}_Y$ is basis for $\mathbb{R}^n$

Since $P(\vec{x}_j) = \vec{x}_j$, $P(\vec{y}_k) = 0$ we have

$$[P]_{\mathcal{B}} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

Let S be std basis for $\mathbb{R}^n$. By Change of Basis Thm

$$[P]_S = B\,[P]_{\mathcal{B}}\,B^{-1}$$

where

$$B = [X|Y] = [\vec{x}_1 \dots \vec{x}_r \,|\, \vec{y}_1 \dots \vec{y}_{n-r}] \text{ is } n \times n.$$

<u>SUMMARY</u> The matrix of P (in std basis) is

$$\boxed{\begin{aligned}[P] &= [X\,|\,Y] \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) [X\,|\,Y]^{-1} \\ &= [X\,|\,0]\,[X\,|\,Y]^{-1}\end{aligned}}$$

<u>EX</u> Let $\{\vec{v}_1 \dots \vec{v}_4\}$ be ONB for $\mathbb{R}^4$

Define $\mathcal{X} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, $\mathcal{Y} = \text{Span}\{\vec{v}_3, \vec{v}_4\}$

So $\mathbb{R}^4 = \mathcal{X} \oplus \mathcal{Y}$

Let $P : \mathbb{R}^4 \to \mathbb{R}^4$ be Proj^n onto $\mathcal{X}$ along $\mathcal{Y}$.

Then

ORTHOGONAL MATRIX

$$[P] = [\vec{v}_1 \;\; \vec{v}_2 \,|\, \vec{0} \;\; \vec{0}]\,[\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3 \;\; \vec{v}_4]^{-1}$$

<u>ASIDE</u>

For a general basis have to invert $[X|Y]$.

$$= \left[\vec{v}_1 \quad \vec{v}_2 \quad \vec{0} \quad \vec{0}\right] \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vec{v}_3^{\,T} \\ \vec{v}_4^{\,T} \end{bmatrix}$$

$$\boxed{P = \vec{v}_1 \vec{v}_1^{\,T} + \vec{v}_2 \vec{v}_2^{\,T}}$$

NOTE

$\perp$ Projn onto Span$\{\vec{v}_1, \vec{v}_2\}$

= Sum of $\perp$ Projns onto Span$(\vec{v}_1)$ and Span$(\vec{v}_2)$.