5.6 UNITARY AND ORTHOGONAL MATRICES

THE ADJOINT OF A L.T.

DEF Let V, W be $\mathbb{R}$ or $\mathbb{C}$ IPS's and $T: V \to W$ a linear transformation.

The ADJOINT of T is the L.T.

$$T^* : W \to V$$

characterized by

$$\boxed{\langle T(\vec{v}) \mid \vec{w} \rangle_W = \langle \vec{v} \mid T^*(\vec{w}) \rangle_V}$$

$$\forall \vec{v} \in V, \ \vec{w} \in W$$

Here $\langle \cdot \mid \cdot \rangle_W$ is the IP on W.

EXAMPLE

Let A be $m \times n$ matrix with $\mathbb{C}$ entries.
Recall

$$A^* := (\overline{A})^T \quad \text{is } n \times m.$$

Define $T: \mathbb{C}^n \to \mathbb{C}^m$ by

$$T(\vec{x}) = A\vec{x}.$$

Equip $\mathbb{C}^n$ with standard IP : $\langle \vec{x}_1 | \vec{x}_2 \rangle_{\mathbb{C}^n} = \vec{x}_1^* \vec{x}_2$.

Then

$$\langle T(\vec{x}) | \vec{y} \rangle_{\mathbb{C}^m} = [T(\vec{x})]^* \vec{y}$$

$$= (A\vec{x})^* \vec{y} = \vec{x}^* (A^* \vec{y})$$

$$= \langle \vec{x} | A^* \vec{y} \rangle_{\mathbb{C}^n}$$

$$= \langle \vec{x} | T^*(\vec{y}) \rangle_{\mathbb{C}^n} \qquad \forall \vec{x} \in \mathbb{C}^n, \ \forall \vec{y} \in \mathbb{C}^m.$$

So we guess $\boxed{T^*(\vec{y}) = A^* \vec{y}}$ should hold.

LEMMA 1 For any L.T. $T : V \to W$ we have

① $\forall \vec{w} \in W \quad \exists \, T^*(\vec{w}) \in V$ so that

$$\langle T(\vec{v}) | \vec{w} \rangle_W = \langle \vec{v} | T^*(\vec{w}) \rangle_V \quad \forall \vec{v} \in V. \qquad (*)$$

② For each $\vec{w} \in W$ $T^*(\vec{w})$ is the ! elt of V so that $(*)$ holds

③ $T^* : W \to V$ is a L.T.

EX Suppose

V has basis $B = \{\vec{v}_1, \vec{v}_2\}$ with

$$M_{ij} = \langle \vec{v}_i | \vec{v}_j \rangle \; , \quad M = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

W has basis $B' = \{\vec{w}_1, \vec{w}_2\}$ with

$$N_{ij} = \langle \vec{w}_i | \vec{w}_j \rangle \; , \quad N = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

and
$$T(\vec{v}_1) = \vec{w}_1 + \vec{w}_2$$
$$T(\vec{v}_2) = \vec{w}_1 - 3\vec{w}_2.$$

So
$$[T]_{BB'} = \left([T(\vec{v}_1)]_{B'} \quad [T(\vec{v}_2)]_{B'} \right) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

Find $[T^*]_{B'B} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Well
$$T^*(\vec{w}_1) = a\vec{v}_1 + b\vec{v}_2$$
$$T^*(\vec{w}_2) = c\vec{v}_1 + d\vec{v}_2$$

$$\langle a\vec{v}_1 + b\vec{v}_2 | \vec{v}_1 \rangle = \langle T^*(\vec{w}_1) | \vec{v}_1 \rangle = \langle \vec{w}_1 | T(\vec{v}_1) \rangle$$

$$a M_{11} + b M_{21} = \langle \vec{w}_1 | \vec{w}_1 + \vec{w}_2 \rangle = N_{11} + N_{12}$$

Continuing in this manner you can set up a system of equations for a, b, c, d which you can solve.

$\boxed{\text{HWK 5.6 AA}}$ ① COMPLETE THIS CALCULATION

② CAN YOU FIND GENERAL FORMULA IF DON'T KNOW ENTRIES OF M AND N.

Pf

① Choose an ONB $\{\vec{u}_1, \dots, \vec{u}_n\}$ for V.

Then $\forall \vec{x} \in V$

$$\vec{x} = \sum_{i=1}^{n} \langle \vec{u}_i \mid \vec{x} \rangle_V \, \vec{u}_i .$$

So if $T^*\vec{w}$ exists it should satisfy

$$T^*(\vec{w}) = \sum_{i=1}^{n} \langle \vec{u}_i \mid T^*(\vec{w}) \rangle_V \, \vec{u}_i$$

$$\overset{*}{=} \sum_{i=1}^{n} \langle T(\vec{u}_i) \mid \vec{w} \rangle_W \, \vec{u}_i \qquad (\dagger)$$

We can use $(\dagger)$ to define $T^*(\vec{w})$.

$\boxed{\text{HWK } 5.6.A}$

Show that if T^* is defined by $(\dagger)$

then $\langle T(\vec{v}) \mid \vec{w} \rangle_W = \langle \vec{v} \mid T^*(\vec{w}) \rangle_V$

must hold for $\forall \vec{v} \in V, \vec{w} \in W$.

<u>POTENTIAL PROBLEM</u>: The value of $T^*(\vec{w})$ could depend on our choice of ONB for V

BUT...

2 Fix $\vec{w} \in W$. Suppose $\vec{v}_1, \vec{v}_2$ are ~~two~~ candidates ~~for~~ "$T^*(\vec{w})$".

Then by $*$ for $j=1$ and $j=2$

$$\langle T(\vec{v}) \mid \vec{w} \rangle_W = \langle \vec{v} \mid \vec{v}_j \rangle \quad \forall \vec{v} \in V.$$

So

$$\langle \vec{v} \mid \vec{v}_2 \rangle_V = \langle T(\vec{v}), \vec{w} \rangle_W = \langle \vec{v} \mid \vec{v}_1 \rangle_V$$

So

$$\langle \vec{v} \mid \vec{v}_2 - \vec{v}_1 \rangle = 0 \quad \forall \vec{v} \in V.$$

Pick $\vec{v} = \vec{v}_2 - \vec{v}_1$ to get $\| \vec{v}_2 - \vec{v}_1 \|^2 = 0$

So $\vec{v}_2 = \vec{v}_1$ as IP is non-degenerate

3 <u>LINEARITY</u>

Suppose that 1, 2 holds.

<u>CLAIM</u> $T^*(\lambda \vec{w}_1 + \vec{w}_2) = \lambda T^*(\vec{w}_1) + T^*(\vec{w}_2)$
$\forall \lambda \in \mathbb{C}$
$\forall \vec{w}_1, \vec{w}_2 \in W$

Pf

$$<\vec{v} \mid T^*(\lambda\vec{w}_1+\vec{w}_2)>_V = <T(\vec{v}) \mid \lambda\vec{w}_1+\vec{w}_2>_W$$

$$= \lambda <T(\vec{v}) \mid \vec{w}_1>_W + <T(\vec{v}) \mid \vec{w}_2>_W$$

$$= \lambda <\vec{v} \mid T^*(\vec{w}_1)>_V + <\vec{v} \mid T^*(\vec{w}_2)>_V$$

$$= <\vec{v} \mid \lambda T^*(\vec{w}_1) + T^*(\vec{w}_2)>_V \qquad \forall \vec{v} \in V$$

So

$$T^*(\lambda_1\vec{w}_1+\vec{w}_2) = \lambda T^*(\vec{w}_1) + T^*(\vec{w}_2) \qquad \text{by !ness}$$

$\boxed{\text{HWK 5.6 B}}$

Use (†) on page 3 to give an alternate proof that T^* is linear.

LEMMA 2

Let V be an IPS with ONB, $\mathcal{B}$ and let $T: V \to V$ be linear.

Then

$$[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*$$

⑥

PF

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an ONB for V.

Since

$$\vec{w} = \sum_{j=1}^{n} \langle \vec{v}_j \mid \vec{w} \rangle \vec{v}_j \qquad \forall \vec{w} \in V$$

we know

$$T(\vec{v}_i) = \sum_{j=1}^{n} \langle \vec{v}_j \mid T(\vec{v}_i) \rangle_V \vec{v}_j$$

Also $$T(\vec{v}_i) = \sum_{j=1}^{n} ([T]_{\mathcal{B}})_{ji} \, \vec{v}_j \quad \text{by def}^n \, [T]_{\mathcal{B}}$$

So by LI of $\mathcal{B}$:

$$([T]_{\mathcal{B}})_{ji} = \langle \vec{v}_j \mid T(\vec{v}_i) \rangle \qquad (**)$$

$$= \overline{\langle T(\vec{v}_i) \mid \vec{v}_j \rangle}$$

$$= \overline{\langle v_i \mid T^*(\vec{v}_j) \rangle} \qquad \text{by } (*)$$

$$= \overline{([T^*]_{\mathcal{B}})_{ij}} \qquad \text{as in } (**)$$

So

$$[T^*]_{\mathcal{B}} = \left(\overline{[T]_{\mathcal{B}}}\right)^T = ([T]_{\mathcal{B}})^*.$$

$\square$

DEF^{ns}

1. $T: V \to V$ is ~~SELF-ADJOINT~~ <u>SELF-ADJOINT</u> IF $T^* = T$.

2. $T: V \to W$ is an <u>ISOMETRY</u> IF

$$\langle T\vec{v}_1 \mid T\vec{v}_2 \rangle_W = \langle \vec{v}_1 \mid \vec{v}_2 \rangle_V \quad \forall \vec{v}_1, \vec{v}_2 \in V$$

<u>NOTE</u> IF $T: \mathbb{R}^n \to \mathbb{R}^n$ is given by $T(\vec{x}) = A\vec{x}$
THEN T SELF-ADJOINT means $A^T = A$,
ie A is SYMMETRIC

<u>PROP 3</u>

1. T ISOMETRY $\Rightarrow$ T is 1-1 ~~and onto~~

2. T ISOMETRY $\Leftrightarrow$ $\|T\vec{v}\|_W = \|\vec{v}\|_V \quad \forall \vec{v} \in V$

(T preserves lengths)

3. IF $T: V \to V$ THEN

$$T \text{ ISOMETRY} \Leftrightarrow T^* \circ T = I_V = T \circ T^*$$
$$\Leftrightarrow T^{-1} = T^*$$

4. IF $T: V \to W$ is an isometry with $\dim W = \dim V$
and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is ONB for V
THEN $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is ONB for W.

PF

1. $T\vec{v} = \vec{0} \Rightarrow \|\vec{v}\| = \|T\vec{v}\| = \vec{0} \Rightarrow \vec{v} = \vec{0}$.

2. $\boxed{\Rightarrow}$ EASY

$\boxed{\Leftarrow}$ Use //gram identity ($U \boxed{\ } V$)

$$\boxed{\langle \vec{x} | \vec{y} \rangle = \frac{1}{4}\left[\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2\right]}$$

So

$$\langle T\vec{v}_1 | T\vec{v}_2 \rangle = \frac{1}{4}\left[\|T\vec{v}_1 + T\vec{v}_2\|^2 - \|T\vec{v}_1 - T\vec{v}_2\|^2\right]$$

$$= \frac{1}{4}\left[\|T(\vec{v}_1+\vec{v}_2)\|^2 - \|T(\vec{v}_1-\vec{v}_2)\|^2\right]$$

$$= \frac{1}{4}\left[\|\vec{v}_1 + \vec{v}_2\|^2 - \|\vec{v}_1 - \vec{v}_2\|^2\right] \quad \text{BY ASSUMPN}$$

$$= \langle \vec{v}_1 | \vec{v}_2 \rangle. \quad \checkmark$$

3. T ISOMETRY

$\Leftrightarrow \langle T\vec{v}_1 | T\vec{v}_2 \rangle = \langle \vec{v}_1 | \vec{v}_2 \rangle \quad \forall \vec{v}_1, \vec{v}_2$

$\Leftrightarrow \langle \vec{v}_1 | T^*T\vec{v}_2 \rangle = \langle \vec{v}_1 | \vec{v}_2 \rangle \quad \forall \vec{v}_1, \vec{v}_2$

$\Leftrightarrow T^*T\vec{v}_2 = \vec{v}_2 \quad \forall \vec{v}_2$

$\Leftrightarrow T^*T = I_V$

NOTE If $T: V \to V$ has $T^*T = I_V$ then T is 1-1

So by Matrix Inversion Thm, T is invertible and $T^{-1} = T^*$

④ $\langle T\vec{v}_i \mid T\vec{v}_j \rangle = \langle \vec{v}_i \mid \vec{v}_j \rangle = \delta_{ij}$ ✓

$\square$

DEF

① $P \in \mathbb{R}^{n\times n}$ is ORTHOGONAL if

$$P^T P = I = P P^T \qquad (P^{-1} = P^T)$$

② $P \in \mathbb{C}^{n\times n}$ is UNITARY if

$$U^* U = I = U U^* \qquad (U^{-1} = U^*)$$

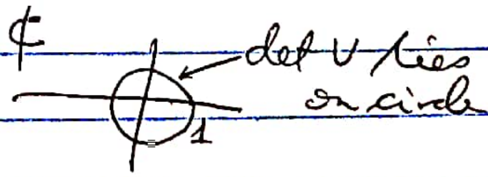
NOTE

ⓐ All orthogonal matrices are unitary

ⓑ $\det P = \pm 1$ as

$$1 = \det I = \det(P^T P) = \det(P^T)\det(P)$$
$$= [\det(P)]^2$$

ⓒ $|\det U| = 1$



as

$$1 = \det U \det U^* = \det U \,\overline{\det U} = |\det U|^2$$

ⓓ ~~Unitary matrices are~~ IF U is UNITARY THEN $T(\vec{x}) = U\vec{x}$ is ISOMETRY of $\mathbb{C}^n$

LEMMA 4

Let $S, T : V \to V$ be linear. Then

$$[T \circ S]_B = [T]_B [S]_B$$

PROP 5

Let V be a $\mathbb{C}$ IPS and $T : V \to V$ an isometry. If B is an ONB for V then $[T]_B$ is unitary.

$\boxed{\text{HWK 5.6C}}$ Prove Lemma 4 and use it to prove Prop 5

PROP 6 Let $U \in \mathbb{C}^{n \times n}$ Then

U is UNITARY $\iff$ The rows (or cols) of U form an ONB for $\mathbb{C}^n$.

NOTE Analogues of PROPS 5+6 hold for orthogonal matrices.

PF If $U = [\vec{u}_1 \dots \vec{u}_n]$ then $U^* = \begin{bmatrix} \vec{u}_1^* \\ \vdots \\ \vec{u}_n^* \end{bmatrix}$ So

$$\langle \vec{u}_i | \vec{u}_j \rangle = \vec{u}_i^* \vec{u}_j = (U^*)_{i*} U_{*j} = (U^* U)_{ij}$$

EXAMPLES OF ORTHOGONAL MATRICES

⓪ I

① Permutation Matrices

A Permⁿ Matrix P is obtained by permuting the rows [or cols] of I.

VB/ CLAIM P has a single $\cancel{\text{zero}}$ 1 in each row + col

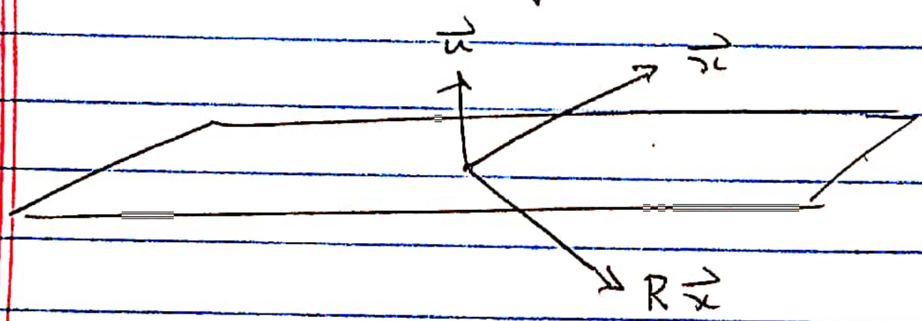
EX $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

P is orthogonal as its rows are same as rows of I and hence form ONB.

② Rotations and Reflections are orthogonal as they preserve lengths of vectors (are isometries)

ⓐ FROM 4.7

Let $\|\vec{u}\| = 1$. Then $R = I - 2\vec{u}\vec{u}^T$ is reflection across plane thru $\vec{0}$ with normal $\vec{u}$



CHECK

$$RR^T = (I - 2uu^T)(I - 2uu^T)^T$$
$$= (I - 2uu^T)(I - 2uu^T) \quad \text{as } (uu^T)^T = uu^T$$
$$= I - 2uu^T - 2uu^T + 4\, u \underbrace{u^T u}_{=1} u^T = I$$

(b) ROTATIONS OF $\mathbb{R}^2$

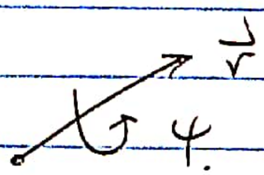
$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{CCW rot}^n \text{ by } \theta$$

$$P^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$
$$= \text{CW rot}^n \text{ by } \theta$$

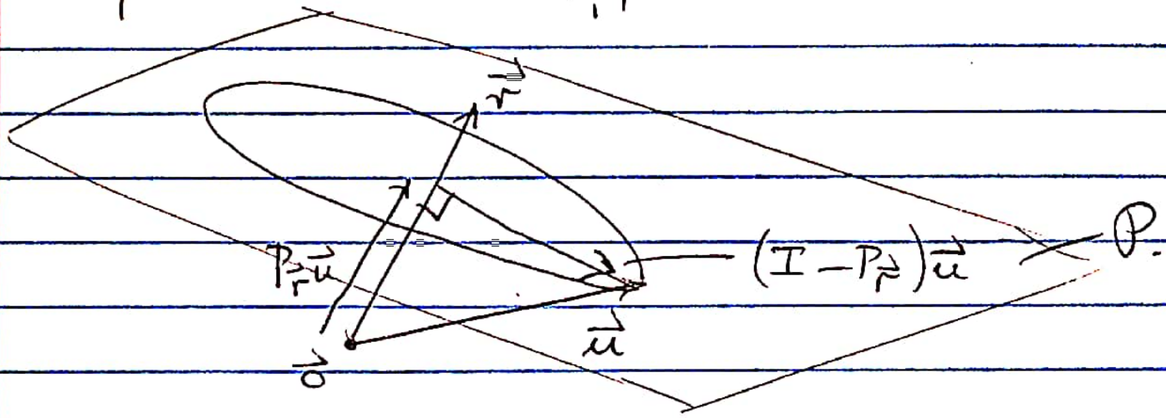
So $P^T P = I = P P^T$

(c) Let $R_{\vec{r},\varphi} : \mathbb{R}^3 \to \mathbb{R}^3$ be rot^n about vector $\vec{r}$ by angle φ

- Choose $\|\vec{r}\| = 1$
- Dir^n of positive angle given by RH rule



Derive a formula for $R_{\vec{r},\psi}$:



Let $P_{\vec{r}} = \vec{r}\,\vec{r}^{\,T} =$ Projn onto Span$(\vec{r})$.

NOW

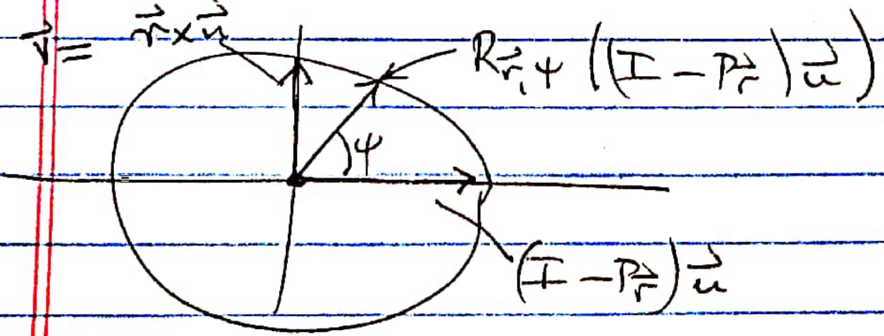
$$\vec{u} = P_{\vec{r}}\,\vec{u} + (I - P_{\vec{r}})\vec{u}$$

By linearity

$$R_{\vec{r},\psi}\,\vec{u} = R_{\vec{r},\psi}\left(P_{\vec{r}}\,\vec{u}\right) + R_{\vec{r},\psi}\left((I - P_{\vec{r}})\vec{u}\right)$$

$$= P_{\vec{r}}\,\vec{u} + R_{\vec{r},\psi}\left((I - P_{\vec{r}})\,\vec{u}\right) \qquad (1)$$

PLANE $\mathcal{P}$ (translated to $\vec{0}$)



$\vec{r}$ = OUT OF PAGE

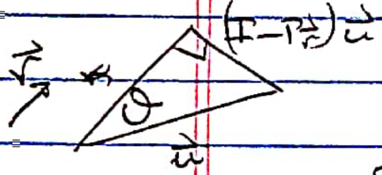
WHY IS $\vec{v} = \vec{r} \times \vec{u}$?

Well (a) $\vec{v} \parallel \vec{r} \times (I - P_{\vec{r}})\vec{u}$

$$= \vec{r} \times \vec{u} - \vec{r} \times (P_{\vec{r}}\vec{u})$$

$$= \vec{r} \times \vec{u} \quad \text{as} \quad P_{\vec{r}}\vec{u} \parallel \vec{r}.$$

(b) $\|(I - P_{\vec{r}})\vec{u}\| = |\vec{u}| \sin\theta$



and $\|\vec{r} \times \vec{u}\| = |\vec{r}|\,|\vec{u}| \sin\theta = |\vec{u}| \sin\theta$

So $\vec{r} \times \vec{u}$ is $90°$ rot^n of $(I - P_{\vec{r}})\vec{u}$

UPSHOT

$$R_{\vec{r},\psi}\left((I - P_{\vec{r}})\vec{u}\right) = \cos(\psi)\,(I - P_{\vec{r}})\vec{u} + \sin(\psi)\,\vec{r} \times \vec{u} \qquad (2)$$

So by (1) + (2)

$$R_{\vec{r},\psi} = (1 - \cos\psi)\,\vec{r}\,\vec{r}^T\vec{u} + \cos\psi\,\vec{u} + \sin\psi\,\vec{r} \times \vec{u}$$

OR

$$R_{\vec{r},\psi} = \cos(\psi)\, I + (1 - \cos\psi)\underbrace{\vec{r}\,\vec{r}^T}_{\text{SYMMETRIC}} + \sin(\psi)\underbrace{\vec{r}\times}_{\text{ANTI-SYMMETRIC}}$$

where $\vec{r}\times$ is the LT

$$(\vec{r}\times)(\vec{u}) = \vec{r}\times\vec{u}$$ which has matrix

$$\vec{r}\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= (r_2u_3 - r_3u_2,\ r_3u_1 - r_1u_3,\ r_1u_2 - r_2u_1)$$

$$= \underbrace{\begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}}_{= \vec{r}\times \text{ ANTISYMMETRIC}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

U ✓

In special case $\vec{r} = \vec{i}$ we get

$$R_{\vec{i},\psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

$=$ Rot^n by ψ about x axis ✓