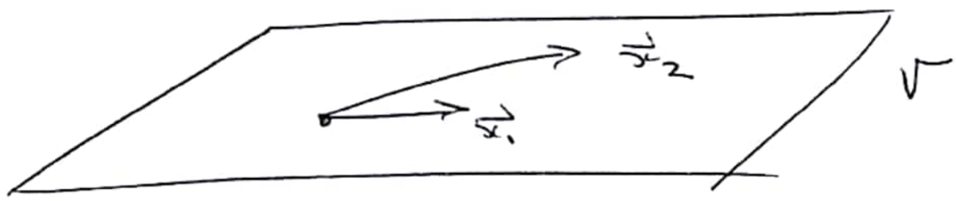
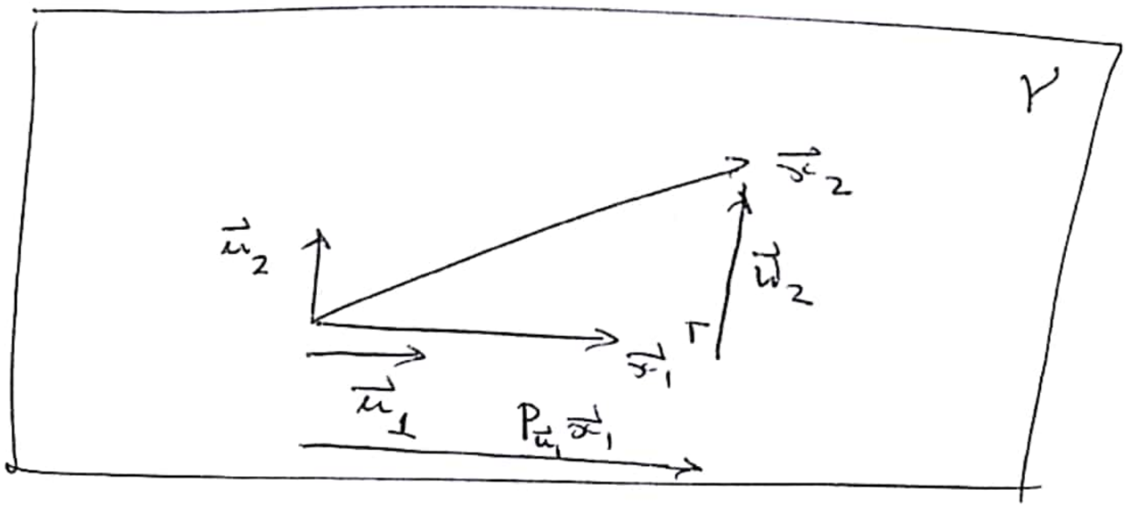
5.5 GRAM-SCHMIDT ORTHOGONALIZATION

MOTIVATING EX

Let $V = \mathrm{Span}\{\vec{x}_1, \vec{x}_2\} \subseteq \mathbb{R}^3$.



GOAL Construct ONB $\mathcal{O} = \{\vec{u}_1, \vec{u}_2\}$ for V.



Set $\vec{u}_1 = \dfrac{\vec{x}_1}{\|\vec{x}_1\|}$

Set $\vec{w}_2 = \vec{x}_2 - P_{\vec{u}_1}\vec{x}_2 = \vec{x}_2 - \langle \vec{x}_2, \vec{u}_1 \rangle \vec{u}_1$

CHECK

$\langle \vec{w}_2 | \vec{u}_1 \rangle = \langle \vec{x}_2 | \vec{u}_1 \rangle - \langle \vec{x}_2 | \vec{u}_1 \rangle \underbrace{\langle \vec{u}_1 | \vec{u}_1 \rangle}_{=1} = 0$

Set $\vec{u}_2 = \vec{w}_2 / \|\vec{w}_2\|$

5.5 GRAM–SCHMIDT ORTHOGONALIZATION

Given a basis

$$B = \{\vec{x}_1, \dots \vec{x}_n\} \quad \text{for IPS } V$$

construct ONB $\mathcal{O} = \{\vec{u}_1, \dots \vec{u}_n\}$ for V using an inductive process:

$\forall k \quad \mathcal{O}_k = \{\vec{u}_1, \dots \vec{u}_k\}$ is ONB for $V_k = \text{Span}\{\vec{x}_1, \dots \vec{x}_k\}$.

$\boxed{k=1} \quad \vec{u}_1 = \dfrac{\vec{x}_1}{\|\vec{x}_1\|}$

INDUCTION STEP

Given $\mathcal{O}_k$ for V_k find $\vec{u}_{k+1}$ so that $\mathcal{O}_{k+1}$ is ONB for V_{k+1}.

If we had $\vec{u}_{k+1}$ we could express $\vec{x}_{k+1}$ as

$$\vec{x}_{k+1} = \sum_{i=1}^{k+1} \langle \vec{u}_i | \vec{x}_{k+1} \rangle \vec{u}_k$$

$$= \sum_{i=1}^{k} \langle \vec{u}_i | \vec{x}_{k+1} \rangle \vec{u}_k + \langle \vec{u}_{k+1} | \vec{x}_{k+1} \rangle \vec{u}_{k+1} \quad (1)$$

(2)

CLAIM

$$\langle \vec{u}_{k+1} | \vec{x}_{k+1} \rangle \neq 0$$

PF

otherwise by (1) $\vec{x}_{k+1} \in \mathrm{Span}(O_k) = \cancel{\mathrm{Span}(V_k)}\ V_k$

contradicting assump^n B is basis for V. $\square$

So by (1)

$$\vec{u}_{k+1} = \frac{\vec{x}_{k+1} - \sum_{i=1}^{k} \langle \vec{u}_i | \vec{x}_k \rangle \vec{u}_i}{\langle \vec{u}_{k+1} | \vec{x}_{k+1} \rangle}$$

We don't care about this denominator as we need $\|\vec{u}_{k+1}\| = 1$

$\boxed{\text{GS ALGORITHM}}$

$$\vec{u}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|}$$

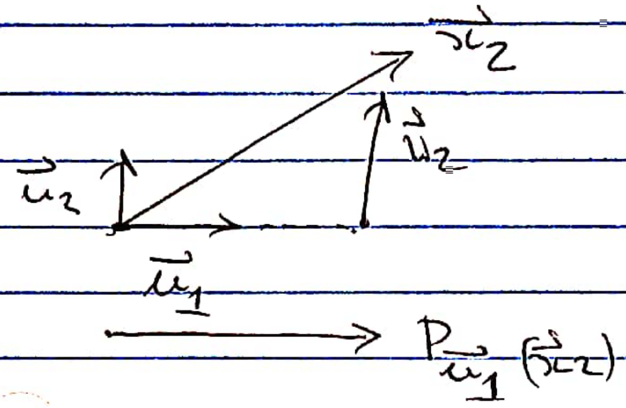
FOR $k > 1$
$$\vec{w}_k = \vec{x}_k - \sum_{i=1}^{k-1} \langle \vec{u}_i | \vec{x}_k \rangle \vec{u}_i$$

$$\boxed{\vec{w}_k = \vec{x}_k - \sum_{i=1}^{k-1} P_{\vec{u}_i}(\vec{x}_k)}$$

$$\vec{u}_k = \frac{\vec{w}_k}{\|\vec{w}_k\|}$$

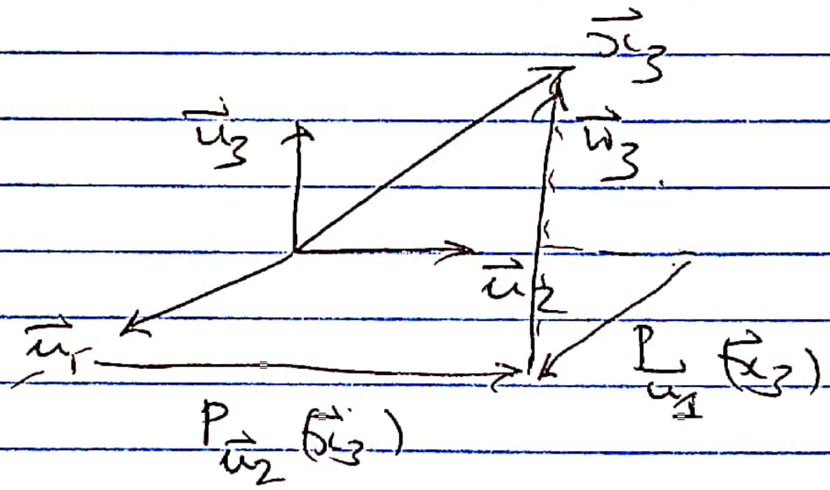
PICTURES

$\boxed{k=2}$



$$\vec{w}_2 = \vec{x}_2 - P_{\vec{u}_1}(\vec{x}_2)$$

$\boxed{k=3}$ $$\vec{w}_3 = \vec{x}_3 - P_{\vec{u}_1}(\vec{x}_3) - P_{\vec{u}_2}(\vec{x}_3)$$



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Ex $\mathbb{R}^3$

1) $\vec{x}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\vec{x}_3 = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$$

2) $\vec{w}_2 = \vec{x}_2 - \langle \vec{u}_1 | \vec{x}_2 \rangle \vec{u}_1$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3/5 & 4/5 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -8/25 \\ 6/25 \\ 3 \end{pmatrix}$$

U ✓ $\langle \vec{w}_2 | \vec{u}_1 \rangle = 0$

$\|\vec{w}_2\| = \sqrt{5725} / 25$

$\vec{u}_2 = \begin{pmatrix} -8 \\ -6 \\ 75 \end{pmatrix} / \sqrt{5725}$.

3) $\vec{w}_3 = \vec{x}_3 - \langle \vec{u}_1 | \vec{x}_3 \rangle \vec{u}_1 - \langle \vec{u}_2 | \vec{x}_3 \rangle \vec{u}_2$

$$= \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} - \frac{1}{25} \begin{pmatrix} 48 \\ 64 \\ 0 \end{pmatrix} - \frac{423}{5725} \begin{pmatrix} -8 \\ 6 \\ 75 \end{pmatrix} = *$$

$\vec{u}_3 = \vec{w}_3 / \|\vec{w}_3\| = \text{ETC}$