5.3, 5.4 INNER PRODUCT SPACES + ORTHOGONAL VECTORS ⓪

MOTIVATING EXS

① DOT PRODUCT ON $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = xu + yv$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2$$

NOTATION $\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = xu + yv$

② AN INNER PRODUCT ON $\mathbb{C}$

$$\mathbb{C} \simeq \mathbb{R}^2$$
$$z \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \qquad z = x + iy$$

$$|z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2.$$

This suggests we define

$$\langle z, w \rangle = \bar{z} w = (x - iy)(u + iv)$$
$$= (xu + yv) + i(xv - yu) \in \mathbb{C}$$

Then $\langle z, z \rangle = \bar{z} z = |z|^2$.

WHAT PROPERTIES ~~SHOULD~~ DOES $\langle z, w \rangle$ HAVE?

5.3, 5.4 INNER PRODUCT SPACES + ORTHOGONAL VECTORS

COMPLEX VECTOR SPACES

Since eigenvalues can be complex, from now on we consider complex vector spaces as well as real ones.

IDEA In a complex vector space the scalars are complex numbers. All axioms of a $\mathbb{C}$ VS are same as for a $\mathbb{R}$ VS except

(M1) $\forall \vec{v} \in V, \ \forall \alpha \in \mathbb{C} \quad \exists$ elmt $\alpha \vec{v} \in V$.

BASIC EX $\quad \mathbb{C}^n = \{ (z_1 \dots z_n) / \ z_j = x_j + i y_j \in \mathbb{C} \}$

INNER PRODUCT SPACES

DEF An INNER PRODUCT on a $\mathbb{R}$ or $\mathbb{C}$ VS V is a mapping

$$\langle \cdot | \cdot \rangle : V \times V \longrightarrow \mathbb{R} \text{ or } \mathbb{C}$$
$$(\vec{x}, \vec{y}) \longmapsto \langle \vec{x} | \vec{y} \rangle \quad \text{so that}$$

① $\langle \vec{x} | \vec{x} \rangle \in \mathbb{R}$ and $\langle \vec{x} | \vec{x} \rangle \geq 0$

with

$$\langle \vec{x} | \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}.$$

2. $\langle \vec{x} | \alpha \vec{y} \rangle = \alpha \langle \vec{x} | \vec{y} \rangle \quad \forall \vec{x}, \vec{y} \in V.$
$\forall \alpha \in \mathbb{R}$ or $\mathbb{C}$

3. $\langle \vec{x} | \vec{y} + \vec{z} \rangle = \langle \vec{x} | \vec{y} \rangle + \langle \vec{x} | \vec{z} \rangle$

4. $\langle \vec{y} | \vec{x} \rangle = \overline{\langle \vec{x} | \vec{y} \rangle}$ COMPLEX CONJUGATE SYMMETRY

NOTES

(a) $\langle \cdot | \cdot \rangle$ is linear in 2ND SLOT

(b) $\langle \cdot | \cdot \rangle$ is conjugate linear in 1st slot
as

$$\langle \alpha \vec{x} | \vec{y} \rangle = \overline{\langle \vec{y} | \alpha \vec{x} \rangle} \quad \text{by } (4)$$

$$= \overline{\alpha \langle \vec{y} | \vec{x} \rangle} \quad \text{by } (2)$$

$$= \bar{\alpha} \, \overline{\langle \vec{y} | \vec{x} \rangle}$$

$$= \bar{\alpha} \langle \vec{x} | \vec{y} \rangle \quad \text{by } (4)$$

DEFS (1) The NORM induced on an Inner Product Space $(V, \langle \cdot | \cdot \rangle)$ is

$$\|\vec{x}\| = \sqrt{\langle \vec{x} | \vec{x} \rangle}$$

THIS IS WHY WE NEED (1).

(2) Let $A \in \mathbb{C}^{n \times m}$
Define

$$A^* = (\overline{A})^T$$

ie $(A^*)_{ij} = \overline{A_{ji}}$

(3) We say A is <u>HERMITIAN SYMMETRIC</u> if $A^* = A$

<u>EXS OF IPS's</u>

(1) $\mathbb{R}^n$ $\langle \vec{x} | \vec{y} \rangle = \underset{\text{ROW} \times \text{COL}}{\vec{x}^T \vec{y}} = \sum_{i=1}^{n} x_i y_i$

(2) $\mathbb{C}^n$ $\langle \vec{x} | \vec{y} \rangle = \vec{x}^* \vec{y} = \sum_{i=1}^{n} \overline{x_i} y_i$

<u>REASONING</u>

Any $\vec{z} \in \mathbb{C}^n$ is of form $\vec{z} = \vec{x} + i\vec{y}$, $\vec{x}, \vec{y} \in \mathbb{R}^n$.
So

$$\|\vec{z}\|^2 = \langle \vec{z} | \vec{z} \rangle$$

$$= \sum_{j=1}^{n} \overline{z_j} z_j = \sum_{j=1}^{n} |z_j|^2$$

$$= \sum_{j=1}^{n} x_j^2 + y_j^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

3 IF A is invertible Then

$$\langle \vec{x} \mid \vec{y} \rangle_A := (A\vec{x})^*(A\vec{y}) = \vec{x}^* A^* A \vec{y}$$

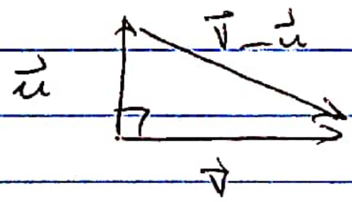
is an IP on $\mathbb{R}^n$ (or $\mathbb{C}^n$)

DEF Two vectors $\vec{x}, \vec{y}$ in an IPS are called ORTHOGONAL if

$$\langle \vec{x} \mid \vec{y} \rangle = 0. \qquad (\vec{x} \perp \vec{y})$$

MOTIVATION FOR DEF FROM Pythagoras:

IN $\mathbb{R}^n$



$$\Leftrightarrow \quad \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{v}-\vec{u}\|^2$$

$$\Leftrightarrow \quad \|\vec{u}\|^2 + \|\vec{v}\|^2 = \langle \vec{v}-\vec{u} \mid \vec{v}-\vec{u} \rangle$$
$$= \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\langle \vec{u} \mid \vec{v} \rangle$$

$$\Leftrightarrow \quad \langle \vec{u} \mid \vec{v} \rangle = 0.$$

DEF

A set of vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ in an IPS V is an **ORTHONORMAL (ON) SET** if $\forall i, j$

$$\langle \vec{u}_i | \vec{u}_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

ie $\quad \vec{u}_i \perp \vec{u}_j \quad$ and $\quad \|\vec{u}_i\| = 1$

LEMMA 1

(a) Every ON Set is LI

(b) Hence every ON Set of n vectors in an n-D VS is a **basis** for V.

We call such bases **ORTHONORMAL BASES**

PF OF (1)

Suppose $\quad \vec{0} = \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n$.

Then

$$\cancel{0 = \langle \vec{0} | \vec{u}_j \rangle = \alpha_1 \langle \vec{u}_1 | \vec{u}_j \rangle + \dots +}$$

$\forall j$

$$0 = \langle \vec{u}_j | \vec{0} \rangle = \alpha_1 \langle \vec{u}_j | \vec{u}_1 \rangle + \dots + \alpha_n \langle \vec{u}_j | \vec{u}_n \rangle$$
$$= \alpha_j$$

$\square$

LEMMA 2 (COEFFICIENTS IN ONB / "FOURIER EXPANSION")

Let $B = \{\vec{u}_1, \ldots, \vec{u}_n\}$ be a ONB for IPS V.

IF $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{u}_i$

THEN

$$\alpha_i = \langle \vec{u}_i | \vec{v} \rangle$$

$\boxed{\text{SURE BEATS GE}}$

ie

$$\boxed{\vec{v} = \sum_{i=1}^{n} \langle \vec{u}_i | \vec{v} \rangle \vec{u}_i}$$

PF

$$\langle \vec{u}_j | \vec{v} \rangle = \langle \vec{u}_j | \sum_i \alpha_i \vec{u}_i \rangle$$

$$= \sum_i \alpha_i \langle \vec{u}_j | \vec{u}_i \rangle$$

$$= \sum_i \alpha_i \delta_{ij} = \alpha_j$$

∎