4.8 CHANGE OF BASIS AND SIMILARITY

A CHANGE OF BASIS MATRIX

Let V be FDVS

Let

$B = \{\vec{x}_1, \dots, \vec{x}_n\}$ "OLD" BASIS

$B' = \{\vec{y}_1, \dots, \vec{y}_n\}$ "NEW" BASIS

DEF THE CHANGE OF BASIS MATRIX FROM BASIS B TO B' IS defined by

$$P := [I]_{BB'} = \left([\vec{x}_1]_{B'}, \dots, [\vec{x}_n]_{B'} \right)$$

So: Cols of P are coord vectors of OLD basis in the NEW basis.

PROP 1 $\forall \vec{v} \in V$ $\boxed{[\vec{v}]_{B'} = P [\vec{v}]_B}$

PF This is just 4.7 PROP 2 with $T = I$:

$$[T(\vec{v})]_{B'} = [T]_{BB'} [\vec{v}]_B$$ $\square$

PROP 2 $\quad P$ is INVERTIBLE and $P^{-1} = ([\vec{y}_1]_{\mathcal{B}}, \ldots, [\vec{y}_n]_{\mathcal{B}})$

PF

Let $Q = [I]_{\mathcal{B}'\mathcal{B}}$ be C of B matrix from New basis back to old

By PROP 1 : $\quad [\vec{v}]_{\mathcal{B}} = Q[\vec{v}]_{\mathcal{B}'}$

So $\forall \vec{v} \in V$

$$[\vec{v}]_{\mathcal{B}} = Q[\vec{v}]_{\mathcal{B}'} = QP[\vec{v}]_{\mathcal{B}}$$

So $\quad QP = I$

Similarly $\quad PQ = I \quad$ So $P^{-1} = Q$ ✓ $\quad \square$

PROP 3 $\quad [\vec{x}_1, \ldots, \vec{x}_n] = [\vec{y}_1, \ldots, \vec{y}_n]P$ when $V = \mathbb{R}^n$. $\quad$ PF 4.7 PROP 3.

$\boxed{B}$ CHANGE OF BASIS THM [Thm 4]

Let $T: V \to V$ and let P be as above

Then

$$[T]_{\mathcal{B}'} = P[T]_{\mathcal{B}}P^{-1}$$

ie If A and B are matrices of a LT T in 2 different bases

Then A and B are similar $\quad$ NOTATION $\quad B \sim A$.

ie $\exists P: B = PAP^{-1}$

NOTE

$$\begin{tikzcd} \mathcal{B}' \arrow[r, "P^{-1}"] \arrow[rrr, bend right, "{[T]_{\mathcal{B}'}}"'] & \mathcal{B} \arrow[r, "{[T]_{\mathcal{B}}}"] & \mathcal{B} \arrow[r, "P"] & \mathcal{B}' \end{tikzcd}$$

PF Let $\vec{v} \in V$

$$P[T]_{\mathcal{B}} P^{-1} [\vec{v}]_{\mathcal{B}'} = P[T]_{\mathcal{B}} Q [\vec{v}]_{\mathcal{B}'}$$

$$= P[T]_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}$$

$$= P[T(\vec{v})]_{\mathcal{B}}$$

$$= [T(\vec{v})]_{\mathcal{B}'}$$

$$= [T]_{\mathcal{B}'} [\vec{v}]_{\mathcal{B}'}$$

Since v is arbitrary $P[T]_{\mathcal{B}} P^{-1} = [T]_{\mathcal{B}'}$ $\square$

EXAMPLE (CONT'D FROM 4.7)

$V = \mathbb{R}^2$

$\mathcal{B} = \{\vec{x}_1, \vec{x}_2\} = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$

$\mathcal{B}' = \{\vec{y}_1, \vec{y}_2\} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$

(a) FIND P by PROP 3

$$[\vec{x}_1 \;\; \vec{x}_2] = [\vec{y}_1, \vec{y}_2] P$$

$$\begin{pmatrix}1 & 3\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} P$$

Solving: $P = \frac{1}{2}\begin{pmatrix}3 & 4\\-1 & 2\end{pmatrix}$ $\quad P^{-1} = \frac{1}{5}\begin{pmatrix}2 & -4\\1 & 3\end{pmatrix}$

LET $T: \mathbb{R}^2 \to \mathbb{R}^2$ be $T(x,y) = (2x+y, 3x-5y)$

From 4.7: $[T]_{\mathcal{B}} = \begin{pmatrix} -5 & 1 \\ 3 & 2 \end{pmatrix}$

So $[T]_{\mathcal{B}'} = P[T]_{\mathcal{B}}P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 9 \\ 5 & 7 \end{pmatrix}$

C INVARIANTS OF L.T.'s

I TRACE

Recall: IF $A \in \mathbb{R}^{n \times n}$ THEN

$$\mathrm{Trace}(A) := \sum_{i=1}^{n} A_{ii}$$

DEF Let $T: V \to V$ and let $\mathcal{B}$ be ANY basis of V.
Define

$$\mathrm{Trace}(T) := \mathrm{Trace}([T]_{\mathcal{B}})$$

PROPS Trace(T) is "well-defied", independent of choice of basis $\mathcal{B}$ for V.

PF By CofB THM, STS

CLAIM IF $B = PAP^{-1}$ THEN $\mathrm{Trace}(B) = \mathrm{Trace}(A)$.

⑤

PF of CLAIM

$\forall$ II ✓ $\quad Tr(PQ) = Tr(QP)$

So

$$Tr(B) = Tr(\underbrace{PA}P^{-1}) = Tr(P^{-1}\underbrace{PA}) = Tr(A) \quad \square$$

II DET

DEF Let $T: V \to V$ and let β be any basis for V. Define

$$\det(T) = \det([T]_\beta)$$

NOTE

$\det(T)$ is well-defined as

$$\det(AB) = \det(A)\det(B).$$

— . —

IMPORTANT IDEA

A special choice of basis can ~~the~~ reveal the structure of a linear transformation or a vector

See

- Hwk 4.8.12
- Spectral Thm in #7
- Discrete Fourier Transform.