4.7 LINEAR TRANSFORMATIONS

DEF $T: U \to V$ is a <u>LINEAR TRANSFORMATION</u> if

$$T(\alpha \vec{x} + \vec{y}) = \alpha T(\vec{x}) + T(\vec{y}) \quad \forall \alpha \in \mathbb{R}, \ \forall \vec{x}, \vec{y} \in U$$

BIG IDEA Many LT's have GEOMETRICAL MEANING

EXS

(0) Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix of real #s. Then

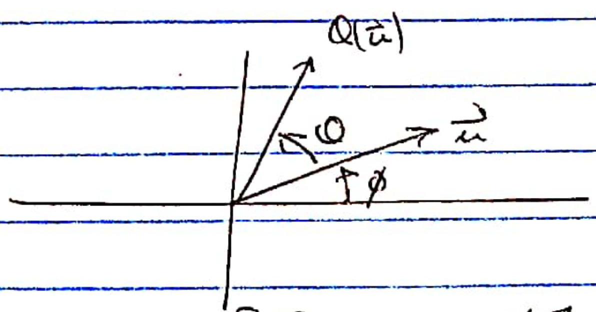
$$T: \mathbb{R}^n \to \mathbb{R}^m \quad \text{defined by}$$

$$T(\vec{x}) = A\vec{x} \quad \text{is a L.T.}$$

"EVERY MATRIX TRANSF^n IS A LINEAR TRANSF^n"

(1) Define $Q: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Q(\vec{u}) = \text{ROTATION of } \vec{u} \text{ CCW by } \theta$$



CLAIM $\exists$ MATRIX $[Q]$ so that

$$Q(\vec{u}) = [Q]\vec{u}.$$

So Q is a L.T.

PF

Write $\vec{u} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix}$ using polar coords

So $Q(\vec{u}) = \begin{pmatrix} R\cos(\phi+\theta) \\ R\sin(\phi+\theta) \end{pmatrix}$

$$= \begin{pmatrix} R\cos\phi\cos\theta - R\sin\phi\sin\theta \\ R\cos\phi\sin\theta + R\sin\phi\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix}$$

$$=: [Q]\,\vec{u}.$$

(2) ORTHOGONAL PROJECTORS

(a) Let $\vec{u} \in \mathbb{R}^n$ have $\|\vec{u}\| = 1$

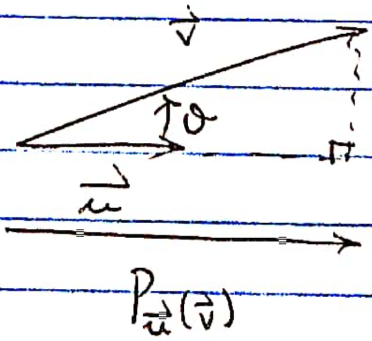
Define

$P_{\vec{u}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be proj^n onto Span$(\vec{u})$

$\|P_{\vec{u}}(\vec{v})\| = \|\vec{v}\|\cos\theta$

$= \vec{u}\cdot\vec{v}$

as $\|\vec{u}\| = 1$



$P_{\vec{u}}(\vec{v}) = \pm\|P_{\vec{u}}(\vec{v})\|\,\vec{u}$

$= \underbrace{(\vec{u}^T\vec{v})}_{1\times 1}\underbrace{\vec{u}}_{n\times 1}$

$= \vec{u}\,(\vec{u}^T\vec{v})$

So $\quad P_{\vec{u}}(\vec{v}) = \underbrace{(\vec{u}\,\vec{u}^T)}_{\substack{(n\times 1)\times(1\times n) \\ n\times n}} \vec{v}$ is linear.

with matrix

$$[P_{\vec{u}}] = \vec{u}\,\vec{u}^T$$

n=2

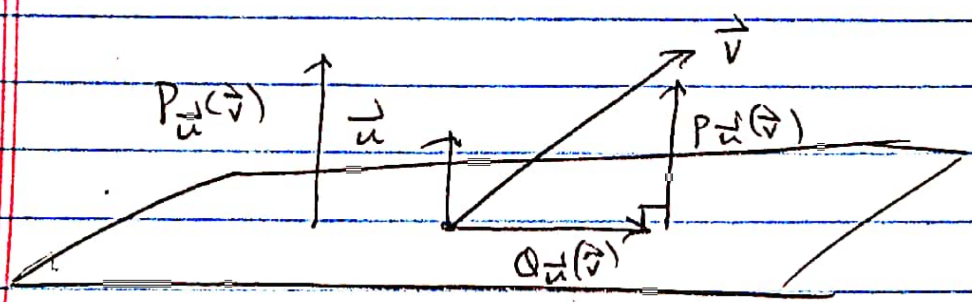
$$[P_{\vec{u}}] = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (u_1 \;\; u_2) = \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix}$$

NOTE $\quad [P_{\vec{u}}]^T = (\vec{u}\,\vec{u}^T)^T = \vec{u}\,\vec{u}^T = [P_{\vec{u}}]$

is symmetric.

b) $Q_{\vec{u}} = I - P_{\vec{u}} : \mathbb{R}^n \to \mathbb{R}^n$ is projection onto

$\vec{u}^\perp$ = Plane thru $\vec{0}$ with normal $\vec{u}$.



$$Q_{\vec{u}}(\vec{v}) = \vec{v} - P_{\vec{u}}(\vec{v})$$

CLAIM

$$Q_{\vec u}(\vec v) \perp P_{\vec u}(\vec v)$$

Pf

Recall

$$\vec x \cdot \vec y = \sum_{j=1}^{n} x_j y_j = (x_1, \dots, x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \vec x^T \vec y$$

So

$$Q_{\vec u}(\vec v) \cdot P_{\vec u}(\vec v) = \left[\vec v - \vec u \vec u^T \vec v\right]^T \vec u \vec u^T \vec v$$

$$= \vec v^T \vec u \, \vec u^T \vec v - \vec v^T \vec u \underbrace{\vec u^T \vec u}\, \vec u^T \vec v$$

$$= (\vec v \cdot \vec u)^2 - (\vec v \cdot \vec u) \underset{\substack{\shortparallel \\ 1}}{\|\vec u\|^2} (\vec u \cdot \vec v) = 0$$

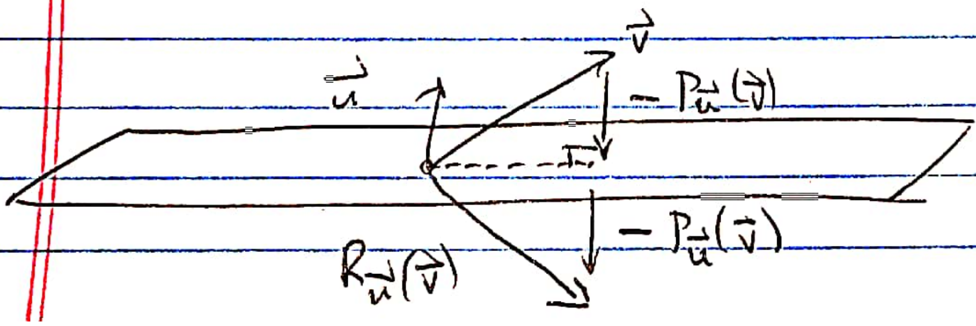
$\square$

3 REFLECTOR

$R_{\vec u} = I - 2P_{\vec u} : \mathbb{R}^n \to \mathbb{R}^n$ is L.T.

that reflects a vector across plane thru $\vec 0$ with normal $\vec u$. Here $\|\vec u\| = 1$ again.

$[R_{\vec u}]^T = [R_{\vec u}]$



MOTIVATING EXAMPLE

P_3 = Polynomials of degree ≤ 3.

Any $p \in P_3$ can be expressed as

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3. \qquad a_j \in \mathbb{R}$$

So the functions $1, x, x^2, x^3$ span P_3

They are also L.I:

If $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$ ← The zero Poly

Then this holds $\forall x \in \mathbb{R}$. So every $x \in \mathbb{R}$ is a zero of p

Since # zeros of nontrivial deg 3 poly is ≤ 3 (FTA)

We conclude p is trivial poly, ie

$$a_0 = a_1 = a_2 = a_3 = 0.$$

So

$B = \{1, x, x^2, x^3\}$ is basis for P_3.

Instead of working with $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ when doing linear algebra we can just work with coefficient/coordinate vector of p w basis:

$$[p]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Scanned with CamScanner

4c

<u>L.T.</u> $\frac{d}{dx} : P_3 \longrightarrow P_3$

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) = a_1 + 2a_2 x + 3a_3 x^2$$

In terms of coordinate vectors

$$\left[\frac{d}{dx}\right]_B \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Matrix of $\frac{d}{dx}$
in basis B.

Scanned with CamScanner

LEMMA 1

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V.

$! =$ UNIQUE

Then $\forall \vec{u} \in V \quad \exists! \ \alpha_1, \dots, \alpha_n \in \mathbb{R}$:

$$\vec{u} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n.$$

PF (a) $\alpha_1 \dots \alpha_n \ \exists$ as $\mathcal{B}$ spans V.

(b) Suppose $\alpha_1 \dots \alpha_n$ and $\beta_1 \dots \beta_n$ are 2 sets of coefficients for $\vec{u}$. Then

$$\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{u} = \beta_1 \vec{v}_1 + \dots + \beta_n \vec{v}_n$$

So $(\alpha_1 - \beta_1)\vec{v}_1 + \dots + (\alpha_n - \beta_n)\vec{v}_n = \vec{0}$

Since $\mathcal{B}$ is LI

$$\alpha_i - \beta_i = 0 \quad \forall i$$
$$\alpha_i = \beta_i$$

So coefficients are !

DEF The COORDINATE VECTOR OF $\vec{u}$ in the basis $\mathcal{B}$ is

$$[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

COOL FORMULA

Suppose $V \leq \mathbb{R}^m$ is given by

$$V = \text{Span}(\mathcal{B}) = \{\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n \,/\, \alpha_j \in \mathbb{R}\}$$

Let

$$A = [\vec{v}_1 \cdots \vec{v}_n] \qquad m \times n$$

Since

$$\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = [\vec{v}_1 \cdots \vec{v}_n]\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{pmatrix}$$ we get

$\forall \vec{u} \in V$ $\quad \boxed{\vec{u} = A[\vec{u}]_{\mathcal{B}}}$

ASIDE $\vec{u} \in V$ $\iff A[\vec{u}]_B = \vec{u}$ is consistent

So $\quad V = \text{Range}(A)$

EX $\quad V = \text{Span}\left\{\begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\\4\end{pmatrix}\right\} \leq \mathbb{R}^4.$

(a) U ✓ $\mathcal{B}$ is a basis.

(b) Let $\vec{u} = \begin{pmatrix}1\\2\\3\\18\end{pmatrix}$. Then to see if $\vec{u} \in V$ and to find $[\vec{u}]_{\mathcal{B}}$ solve $A[\vec{u}]_{\mathcal{B}} = \vec{u}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 18 \end{pmatrix}$$

G.E. gives $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = [\vec{u}]_B$

So $\vec{u} = -\vec{v}_1 + 2\vec{v}_2 + 4\vec{v}_3$.

— o —

$\boxed{\text{HWK 4.7 A}}$ Show that $\forall \lambda_j \in \mathbb{R}, \vec{v}_j \in V$

$$[\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2]_B = \lambda_1 [\vec{v}_1]_B + \lambda_2 [\vec{v}_2]_B$$

— o —

THE MATRIX OF A L.T.

DEF

Let $T: U \to V$ be a L.T.

Let $B = \{\vec{u}_1, \dots \vec{u}_n\}$ be a basis for U

$B' = \{\vec{v}_1, \dots \vec{v}_m\}$ —————— V

Since $T(\vec{u}_j) \in V$ it has coords wrt B'

NOTE ORDER OF INDICES IN α_{ij}

$$T(\vec{u}_j) = \sum_{i=1}^{m} \alpha_{ij} \vec{v}_i$$

So

$$\alpha_{*j} = [T(\vec{u}_j)]_{B'} \qquad \text{COLUMN } j$$

The MATRIX of T wrt bases B and B' is

$$[T]_{BB'} = [T] = \left([T(\vec{u}_1)]_{B'}, \ldots, [T(\vec{u}_n)]_{B'} \right)$$

So

$$[T]_{ij} = \alpha_{ij}$$

NOTE IF $T : U^n \to V^m$

THEN $[T]$ is $m \times n$.

PROP 2

$$[T(\vec{u})]_{B'} = [T]_{BB'} [\vec{u}]_B$$

$m \times 1$ $\qquad$ $m \times n$ $\qquad$ $n \times 1$

PF Let $\vec{u} = \sum_{i=1}^{n} \alpha_i \vec{u}_i$

So $T(\vec{u}) = \sum_{i=1}^{n} \alpha_i T(\vec{u}_i)$

By HWK 4.7A

$$[T(\vec{u})]_{B'} = \sum_{i=1}^{n} \alpha_i [T(\vec{u}_i)]_{B'}$$

$$= \left([T(\vec{u}_1)]_{B'}, \dots, [T(\vec{u}_n)]_{B'}\right) \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$$

$$= [T]_{BB'} [\vec{u}]_B \qquad \square$$

PROP 3 Let $U = \mathbb{R}^n$, $V = \mathbb{R}^m$

Then

$$\underbrace{[T(\vec{u}_1), \dots, T(\vec{u}_n)]}_{m \times n} = \underbrace{[\vec{v}_1 \dots \vec{v}_m]}_{m \times m} \underbrace{[T]_{BB'}}_{m \times n}$$

PF

$$T(\vec{u}_i) = [\vec{v}_1 \dots \vec{v}_m] [T(\vec{u}_i)]_{B'} \quad \text{by PROP 2}$$

So

$$[T(\vec{u}_1) \dots T(\vec{u}_n)] = [v_1 \dots v_m] \left([T(\vec{u}_1)]_{B'}, \dots, [T(\vec{u}_n)]_{B'}\right)$$

$$= [\vec{v}_1 \dots \vec{v}_m] [T]_{BB'} \quad \text{by DEF}^{n}. \qquad \square$$

EX Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be

$$T(x,y) = (2x+y,\ 3x-5y)$$

Let $B = B' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$

Find $[T] = [T]_{BB'}$,

Well $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ MATRIX OF T IN STD BASIS

So
$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}, \quad T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

PROP 3 gives
$$[T(\vec{u}_1),\ T(\vec{u}_2)] = [\vec{u}_1,\ \vec{u}_2]\,[T]$$

$$\begin{bmatrix} 4 & 7 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

Use GE we get

$$[T] = \begin{pmatrix} -5 & 1 \\ 3 & 2 \end{pmatrix}$$

~~AND~~ ~~by~~ so $T(\vec{u}_j) = \sum_{i=1}^{m} \alpha_{ij} v_i$ becomes

$$\begin{pmatrix} 4 \\ -7 \end{pmatrix} = -5\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$